Ex libris
universitatis
albertaensis

RKColler

PROVINCIAL ARCHIVES
OF ALBERTA
ACC. 70.348

Digitized by the Internet Archive
in 2016

https://archive.org/details/elementaryalgebrsmit

WORKS BY CHARLES SMITH, M.A.

ELEMENTARY ALGEBRA. Revised and Enlarged. Globe 8vo. 4*s.* 6*d.*

⁂ **KEY.** Crown 8vo. 10*s.* 6*d.*

A TREATISE ON ALGEBRA. Crown 8vo. 7*s.* 6*d.*

⁂ **KEY.** Crown 8vo. 10*s.* 6*d.*

AN ELEMENTARY TREATISE ON CONIC SECTIONS. Crown 8vo. 7*s.* 6*d.*

⁂ **KEY.** Crown 8vo. 10*s.* 6*d.*

AN ELEMENTARY TREATISE ON SOLID GEOMETRY. Crown 8vo. 9*s.* 6*d.*

GEOMETRICAL CONICS. Crown 8vo. 6*s.*

⁂ **KEY.** Crown 8vo. 6*s.*

By CHARLES SMITH, M.A. & SOPHIE BRYANT, D.Sc.

EUCLID'S ELEMENTS OF GEOMETRY. Globe 8vo. Books I.—IV., VI. and XI. 4*s.* 6*d.* Book I. 1*s.* Books I. and II. 1*s.* 6*d.* Books III. and IV. 2*s.* Books I.—IV. 3*s.* Books VI. and XI. 1*s.* 6*d.* KEY. Crown 8vo. 8*s.* 6*d.*

MACMILLAN AND CO., Ltd., LONDON.

SOLUTIONS OF THE EXAMPLES

IN

CHARLES SMITH'S

ELEMENTARY ALGEBRA.

MACMILLAN AND CO., Limited
LONDON · BOMBAY · CALCUTTA
MELBOURNE

THE MACMILLAN COMPANY
NEW YORK · BOSTON · CHICAGO
ATLANTA · SAN FRANCISCO

THE MACMILLAN CO. OF CANADA, Ltd.
TORONTO

SOLUTIONS OF THE EXAMPLES

IN

CHARLES SMITH'S

ELEMENTARY ALGEBRA

BY

A. G. CRACKNELL, B.A.,
LATE SCHOLAR OF SIDNEY SUSSEX COLLEGE, CAMBRIDGE;
MATHEMATICAL MASTER OF SUNBURY MILITARY COLLEGE.

MACMILLAN AND CO., LIMITED
ST. MARTIN'S STREET, LONDON
1910

First Edition 1891.
Reprinted with additions 1901, 1904, 1910.

LIBRARY
UNIVERSITY OF ALBERTA

PREFACE.

In dealing with each example in this work it has been my object to give the solution which follows most naturally from the formulae and theorems with which the Student is acquainted at that stage.

I have also taken great care that the solutions should be sufficiently clear and explicit to be of use to Students who are reading the subject without the aid of a Teacher.

I have to express my deepest thanks to Mr Smith for the very valuable help which he has afforded me, and for his revision of the sheets for the Press.

A. G. CRACKNELL.

SUNBURY MILITARY COLLEGE,
June, 1891.

2628918

KEY TO ELEMENTARY ALGEBRA.

I. PAGE 3.

1. $7+6+4=13+4=17.$ **2.** $5-3+4=2+4=6.$

3. $11+7-12-6=18-12-6=6-6=0.$ **4.** $7\times6\times4=42\times4=168.$

5. $5\div3\times4=\frac{5}{3}\times4=\frac{20}{3}.$ **6.** $11\times7\div12\div6=77\div12\div6=\frac{77}{12}\div6=\frac{77}{72}.$

7. $3-2=1.$ **8.** $4-1=3.$ **9.** $7\times1-3\times2=1.$ **10.** $10\times2-6\times3=2.$

11. $5\times1-2\times2+6\times3-4\times4=5-4+18-16=3.$

12. $13\times1-6\times2+7\times3-5\times4=13-12+21-20=2.$

13. $18\times2-3\times3-4\times4+9\times1=36-9-16+9=20.$

14. $20\times1\times2-3\times3\times4=40-36=4.$ **15.** $4\times4\times1-2\times2\times3=16-12=4.$

16. $1\times2\times3+2\times3\times4+3\times4\times1+4\times1\times2=6+24+12+8=50.$

17. $3\times6\times5+2\times2\times5+5\times6=90+20+30=140.$

18. $7\times6\times0+9\times2\times5-3\times5\times6=0+90-90=0.$

19. $6\times5\div2=30\div2=15.$ **20.** $6\div5\times2=\frac{6}{5}\times2=\frac{12}{5}.$

21. $2\times5\div6\div2=10\div6\div2=\frac{10}{6}\div2=\frac{5}{6}.$

II. PAGES 7—8.

13. $(10+15)(9-6)=25\times3=75.$ **14.** $(5+6)(1+0)=11\times1=11.$

15. $(15-12)^3-2(9-6)^2+2(0+3)^2=3^3-2\times3^2+2\times3^2=27.$

16. $4\times5^3+4\times3^3+4\times1^3-3(3+1)(1+5)(5+3)$
$=4\times125+4\times27+4\times1-3\times4\times6\times8=36.$

17. $5(5+1)^3(3-1)^2-\frac{1}{25}(5-0)^3(3+3)^2=5\times6^3\times2^2-\frac{1}{25}\times5^3\times6^2$
$=4320-180=4140.$

18. When $x=2$, $x^2-5x+6=4-10+6$, and when $x=3$,
$x^2-5x+6=9-15+6.$

19. (i) $2\times 8-11\times 4+17\times 2-6=16-44+34-6=0.$

(ii) $2\times 27-11\times 9+17\times 3-6=54-99+51-6=0.$

(iii) $2\times\frac{1}{8}-11\times\frac{1}{4}+17\times\frac{1}{2}-6=\frac{1}{4}-\frac{11}{4}+\frac{17}{2}-6=0.$

20. $\sqrt{(25-16)}=\sqrt{9}=3.$ **21.** $\sqrt{5\times 5}=5.$

22. $\sqrt{2\times 4\times\frac{1}{2}+3\times 5}=\sqrt{4+15}=17.$ **23.** $\sqrt[3]{4\times\frac{1}{2}+5^2}=\sqrt[3]{27}=3.$

24. $\sqrt[3]{\left(2\times 25+16-8\times\frac{1}{4}\right)}=\sqrt[3]{(50+16-2)}=\sqrt[3]{64}=4.$

25. $\sqrt[3]{\left(4\times 25-\frac{1}{2}\times 16+\frac{1}{2}\times\frac{1}{4}-1\right)}=\sqrt[3]{\frac{729}{8}}=\frac{9}{2}.$

26. $\sqrt{(5+4)}\times\sqrt[3]{\left(3\times 5\times 4+2\times 4\times\frac{1}{2}\right)}=\sqrt{9}\times\sqrt[3]{64}=3\times 4.$

27. $(6+4)^2(5+8)^2-4(6\times 5+4\times 8)(4\times 5+6\times 8)$
$$=10^2\times 13^2-4\times 62\times 68=16900-16864=36.$$

28. $\sqrt{18(18-9)(18-12)(18-15)}=\sqrt{18\times 9\times 6\times 3}=\sqrt{54^2}=54.$

29. $2s=9+12+15=36;\ \therefore s=18.$ Then
$$\sqrt{\frac{s(s-a)}{(s-b)(s-c)}}=\sqrt{\frac{18\times 9}{6\times 3}}=\sqrt{9}=3.$$

30. $\sqrt{(450+1152+3200-4096-625-81)}=\sqrt{(4802-4802)}=0.$

31. (i) each $=27$, (ii) each $=65$, (iii) each $=95$.

32. (i) each $=19$, (ii) each $=189$, (iii) each $=117$.

IV. PAGE 15.

1. $2a+3a+5a+b+2b+3c+2c+c=10a+3b+6c.$

2. $6a-4a-3b+2b+2c-3c=2a-b-c.$

3. $5a^2-4a^2-3a+6a+6+3=a^2+3a+9.$

4. $7a^3-3a^3-3a^2-4a+2a+9+7-16=4a^3-3a^2-2a.$

5. $5a^3-5a^3-4a^2b-4a^2b+3ab^2-3b^3=-8a^2b+3ab^2-3b^3.$

6. $3a^2-2a^2-a^2+6ab-4ab-2ab-4b^2+3b^2+b^2=0.$

7. $\begin{array}{r} a+b \\ a-b \\ \hline 2a \end{array}$ **8.** $\begin{array}{r} 2x-y \\ 2x+y \\ \hline 4x \end{array}$ **9.** $\begin{array}{r} \frac{1}{2}a+\frac{1}{2}b \\ -\frac{1}{2}a+\frac{1}{2}b \\ \hline b \end{array}$ **10.** $\begin{array}{r} \frac{1}{3}a+\frac{4}{3}b \\ -\frac{1}{3}a+\frac{2}{3}b \\ \hline 2b \end{array}$

11.
$$\begin{array}{r} a^2-a \\ +a+a^3 \\ \hline a^2 \quad +a^3 \end{array}$$

12.
$$\begin{array}{r} a+a^2+4a^3 \\ -4a-a^2+2a^3 \\ \hline -3a \qquad +6a^3 \end{array}$$

13.
$$\begin{array}{r} m^2+mn+n^2 \\ m^2-mn-n^2 \\ \hline 2m^2 \qquad\qquad \end{array}$$

14.
$$\begin{array}{r} 3p^2+5pq-6q^2 \\ -3p^2-4pq+5q^2 \\ \hline pq-q^2 \end{array}$$

15.
$$\begin{array}{r} 3a^2-2ab+b^2 \\ a^2-2ab-\frac{3}{2}b^2 \\ \hline 4a^2-4ab-\frac{1}{2}b^2 \end{array}$$

16.
$$\begin{array}{r} 2a+b-3c \\ -3a+2b+c \\ a-3b+2c \\ \hline 0 \qquad \end{array}$$

17.
$$\begin{array}{r} 4a-3b-c \\ -a+4b-3c \\ -3a-b+4c \\ \hline 0 \qquad \end{array}$$

18.
$$\begin{array}{r} 4a^2-3ab+b^2 \\ a^2+4ab-3b^2 \\ -3a^2+ab+4b^2 \\ \hline 2a^2+2ab+2b^2 \end{array}$$

19.
$$\begin{array}{r} a-\frac{1}{2}b+\frac{1}{2}c \\ \frac{1}{2}a+b-\frac{1}{2}c \\ -\frac{1}{2}a+\frac{1}{2}b+c \\ \hline a+b+c \end{array}$$

20.
$$\begin{array}{r} 4x-2y+1 \\ -3x-y+2 \\ x+3y-3 \\ \hline 2x \qquad\qquad \end{array}$$

21.
$$\begin{array}{r} -4a-b+2 \\ 8a-5b+2 \\ -a-4b-2 \\ \hline 3a-10b+2 \end{array}$$

22.
$$\begin{array}{r} x^3-2x^2y-2xy^2 \qquad \\ x^2y-3xy^2-y^3 \\ -x^3 \qquad +3xy^2-2y^3 \\ \hline -x^2y-2xy^2-3y^3 \end{array}$$

23.
$$\begin{array}{r} a^3+4b^3-5c^3+3abc \\ -5a^3+b^3+4c^3+6abc \\ 4a^3-5b^3+c^3-9abc \\ \hline 0 \qquad\qquad \end{array}$$

24.
$$\begin{array}{r} 5a^3-2a^2b+9ab^2+17b^3 \\ -2a^3+5a^2b-4ab^2-12b^3 \\ -a^3-5a^2b-4ab^2+b^3 \\ -2a^3+2a^2b-ab^2-6b^3 \\ \hline 0 \qquad\qquad \end{array}$$

25.
$$\begin{array}{r} \frac{2}{3}a^3-\frac{4}{5}a^2b+3b^3 \qquad\qquad \\ a^3 \qquad -\frac{3}{2}b^3-\frac{2}{3}ab^2 \\ \frac{1}{3}a^2b-\frac{1}{6}b^3-\frac{2}{5}ab^2 \\ \hline \frac{5}{3}a^3-\frac{7}{15}a^2b+\frac{4}{3}b^3-\frac{16}{15}ab^2 \end{array}$$

26.
$$\begin{array}{r} 3x^3 \qquad -4x+5 \\ 2x^2-6x+7 \\ 6x^3-2x^2-2x \qquad \\ -4x^3 \qquad +8x+3 \\ \hline 5x^3 \qquad -4x+15 \end{array}$$

27.
$$\begin{array}{r} x^3-3ax^2+5a^2x-a^3 \\ 2x^3+4ax^2-6a^2x \qquad \\ 6ax^2-3a^2x+a^3 \\ -2x^3 \qquad +4a^2x-5a^3 \\ \hline x^3+7ax^2 \qquad -5a^3 \end{array}$$

28.
$$\begin{array}{r} 3x^2+y^2-3yz-z^2 \qquad \\ -3y^2+3yz \qquad +2xy \\ -4x^2+y^2 \qquad +z^2-2xy \\ \hline -x^2-y^2 \qquad\qquad\qquad \end{array}$$

29.
$$\begin{array}{r} b+2c-3a \\ -3b+c+2a \\ +2b-3c+a \\ \hline 0 \qquad \end{array}$$

30.
$$\begin{array}{r} 5x-3y-2z \\ -2x+5y-3z \\ -3x-2y+5z \\ \hline 0 \qquad \end{array}$$

V. Pages 19—20.

19. Subtract $2a-3b$ from $4a-6b$.

20. Subtract a^2+b^2 from $2a^2-ab$.

21. Subtract $5ab-2bc+3ca$ from $7ab+2ca$.

22. Subtract $a^2+3b^2+2c^2$ from b^2-3a^2.

23, 24, 25. Change the signs of all the terms in the three expressions to be subtracted, and then add to the fourth expression.

VI. Page 22.

1. $a+b-a+b=2b$.

2. $a-b-a-b=-2b$.

3. $a-b-c+b-c-a=-2c$.

4. $3x-y+2x+z+y-5x=z$.

5. $x-y+z-x=-y+z$.

6. $a-[a-\{a-a\}]=a-[a]=0$.

7. $1-[2-\{3-4+5\}]=1-[2-3+4-5]=1-2+3-4+5=3$.

8. $$a+b-[a-b+\{a+b-a+b\}]=a+b-[a-b+a+b-a+b]$$
$$=a+b-a+b-a-b+a-b=0.$$

9. $$5-[4+\{5-(4+5-4)\}]=5-[4+\{5-4-5+4\}]=5-[4+5-4-5+4]$$
$$=5-4-5+4+5-4=1.$$

10. $$x-[y-\{z-(x-y+z)\}]=x-[y-\{z-x+y-z\}]=x-[y-z+x-y+z]$$
$$=x-y+z-x+y-z=0.$$

11. $3x-\{2y+5z-3x-y\}=3x-2y-5z+3x+y=6x-y-5z$.

12. $$\{2x-(5y-3z-7)\}-[4+\{x-3y-2z-5\}]$$
$$=\{2x-5y+3z+7\}-[4+x-3y-2z-5]$$
$$=2x-5y+3z+7-4-x+3y+2z+5=x-2y+5z+8.$$

13. $$[2a-\{3b+(4c-3b-2a)\}]=[2a-\{3b+4c-3b-2a\}]$$
$$=[2a-3b-4c+3b+2a]=4a-4c.$$

14. $x-y+z+\{2z-3y+5x\}=x-y+z+2z-3y+5x=6x-4y+3z$.

15. $a-2b-\{3a-b+c-5c\}=a-2b-3a+b-c+5c=-2a-b+4c$.

16. $$a-[3b+\{3c-d+b+a\}-2a]=a-[3b+3c-d+b+a-2a]$$
$$=a-3b-3c+d-b-a+2a=2a-4b-3c+d.$$

17. $$3a-[2b-\{4c-12a-4b+8c\}-6b+12c]$$
$$=3a-[2b-4c+12a+4b-8c-6b+12c]$$
$$=3a-2b+4c-12a-4b+8c+6b-12c=-9a.$$

18. $$\{2x-3y+7z+3x-2y+9z\}-\{y-5z-3x+y+2z+8z\}$$
$$=2x-3y+7z+3x-2y+9z-y+5z+3x-y-2z-8z=8x-7y+11z.$$

19. $a^2-3ab+4b^2-2a^2+3ab-6b^2-\{5b^2-(3ab-7a^2+b^2)\}$

$=a^2-3ab+4b^2-2a^2+3ab-6b^2-\{5b^2-3ab+7a^2-b^2\}$

$=a^2-3ab+4b^2-2a^2+3ab-6b^2-5b^2+3ab-7a^2+b^2$

$=-8a^2+3ab-6b^2.$

20. $m^2-n^2-\{3mn-5n^2+m^2\}+[n^2-\{3mn-5m^2+6n^2\}+8mn]$

$=m^2-n^2-3mn+5n^2-m^2+[n^2-3mn+5m^2-6n^2+8mn]$

$=m^2-n^2-3mn+5n^2-m^2+n^2-3mn+5m^2-6n^2+8mn$

$=5m^2-n^2+2mn.$

VIII. PAGE 31.

20. $2\{3ab-4ac+8ab\}=6ab-8ac+16ab=22ab-8ac.$

21. $3a-2[b-\{2c-6a-2b+4c\}-3b+6c]$

$=3a-2[b-2c+6a+2b-4c-3b+6c]$

$=3a-2b+4c-12a-4b+8c+6b-12c=-9a.$

22. $4a^3-[2b^3-3c^3-6b^2c-6bc^2+3c^3+6b^2c]$

$=4a^3-2b^3+3c^3+6b^2c+6bc^2-3c^3-6b^2c=4a^3-2b^3+6bc^2.$

23. $7ac-2\{2ac-6bc-15ac+6ab\}$

$=7ac-4ac+12bc+30ac-12ab=33ac+12bc-12ab.$

IX. PAGE 34.

1.
$$\begin{array}{lll} x & +2y & \\ x & -2y & \\ \hline x^2 & +2xy & \\ & -2xy & -4y^2 \\ \hline x^2 & & -4y^2 \end{array}$$

2.
$$\begin{array}{lll} a & -3b & \\ a & +3b & \\ \hline a^2 & -3ab & \\ & +3ab & -9b^2 \\ \hline a^2 & & -9b^2 \end{array}$$

3.
$$\begin{array}{lll} 2x & +3y & \\ 3x & -2y & \\ \hline 6x^2 & +9xy & \\ & -4xy & -6y^2 \\ \hline 6x^2 & +5xy & -6y^2 \end{array}$$

4.
$$\begin{array}{lll} 5a & +4b & \\ a & -b & \\ \hline 5a^2 & +4ab & \\ & -5ab & -4b^2 \\ \hline 5a^2 & -ab & -4b^2 \end{array}$$

5.
$$\begin{array}{lll} x & +7 & \\ x & +6 & \\ \hline x^2 & +7x & \\ & 6x & +42 \\ \hline x^2 & +13x & +42 \end{array}$$

6.
$$\begin{array}{lll} x & -7 & \\ x & -6 & \\ \hline x^2 & -7x & \\ & -6x & +42 \\ \hline x^2 & -13x & +42 \end{array}$$

7.
$$\begin{array}{lll} x & +7 & \\ x & -6 & \\ \hline x^2 & +7x & \\ & -6x & -42 \\ \hline x^2 & +x & -42 \end{array}$$

8.
$$\begin{array}{lll} a & +9 & \\ a & -5 & \\ \hline a^2 & +9a & \\ & -5a & -45 \\ \hline a^2 & +4a & -45 \end{array}$$

9.
$$\begin{array}{lll} 2x & -4 & \\ 2x & +6 & \\ \hline 4x^2 & -8x & \\ & +12x & -24 \\ \hline 4x^2 & +4x & -24 \end{array}$$

10.
$$\begin{array}{rrr} & 3x & -\ 7\\ & 2x & -\ 1\\ \hline 6x^2 & -\ 14x & \\ & -\ 3x & +7\\ \hline 6x^2 & -\ 17x & +7\end{array}$$

11.
$$\begin{array}{rrr} & 2y & +\ 5b\\ & 3y & -\ 4b\\ \hline 6y^2 & +15by & \\ & -\ 8by & -20b^2\\ \hline 6y^2 & +\ 7by & -20b^2\end{array}$$

12.
$$\begin{array}{rrr} & 3m^2 & -1\\ & 3m^2 & +1\\ \hline 9m^4 & -3m^2 & \\ & +3m^2 & -1\\ \hline 9m^4 & & -1\end{array}$$

13.
$$\begin{array}{rrr} & 2m^2 & +\ 5n^2\\ & 2m^2 & -\ 5n^2\\ \hline 4m^4 & +10m^2n^2 & \\ & -10m^2n^2 & -25n^4\\ \hline 4m^4 & & -25n^4\end{array}$$

14.
$$\begin{array}{rrr} & a & +\frac{1}{2}b\\ & a & -\frac{1}{2}b\\ \hline a^2 & +\frac{1}{2}ab & \\ & -\frac{1}{2}ab & -\frac{1}{4}b^2\\ \hline a^2 & & -\frac{1}{4}b^2\end{array}$$

15.
$$\begin{array}{rrr} & 2a & +\frac{1}{3}b\\ & 3a & +\frac{1}{2}b\\ \hline 6a^2 & +\ ab & \\ & +\ ab & +\frac{1}{6}b^2\\ \hline 6a^2 & +2ab & +\frac{1}{6}b^2\end{array}$$

16.
$$\begin{array}{rrr} & \frac{1}{4}a & -\ \frac{1}{3}\ b\\ & \frac{1}{3}a & -\ \frac{1}{4}\ b\\ \hline \frac{1}{12}a^2 & -\ \frac{1}{9}\ ab & \\ & -\ \frac{1}{16}\ ab & +\frac{1}{12}b^2\\ \hline \frac{1}{12}a^2 & -\frac{25}{144}ab & +\frac{1}{12}b^2\end{array}$$

17.
$$\begin{array}{rrrr} & x^2 & +\ x & +1\\ & & x & -1\\ \hline x^3 & +\ x^2 & +\ x & \\ & -\ x^2 & -\ x & -1\\ \hline x^3 & & & -1\end{array}$$

18.
$$\begin{array}{rrrr} & x^2 & -\ x & +1\\ & & x & +1\\ \hline x^3 & -\ x^2 & +\ x & \\ & +\ x^2 & -\ x & +1\\ \hline x^3 & & & +1\end{array}$$

19.
$$\begin{array}{rrrr} & a^2 & +ab & +b^2\\ & & a & -\ b\\ \hline a^3 & +a^2b & +ab^2 & \\ & -a^2b & -ab^2 & -b^3\\ \hline a^3 & & & -b^3\end{array}$$

20.
$$\begin{array}{rrrr} & a^2 & -ab & +b^2\\ & & a & +\ b\\ \hline a^3 & -a^2b & +ab^2 & \\ & +a^2b & -ab^2 & +b^3\\ \hline a^3 & & & +b^3\end{array}$$

21.
$$\begin{array}{rrrr} & 4a^2 & +\ 6ab & +\ 9b^2\\ & & 2a & -\ 3b\\ \hline 8a^3 & +12a^2b & +18ab^2 & \\ & -12a^2b & -18ab^2 & -27b^3\\ \hline 8a^3 & & & -27b^3\end{array}$$

22.
$$\begin{array}{rrrr} & 16p^2 & +20pq & +\ 25q^2\\ & & 4p & -\ 5q\\ \hline 64p^3 & +80p^2q & +100pq^2 & \\ & -80p^2q & -100pq^2 & -125q^3\\ \hline 64p^3 & & & -125q^3\end{array}$$

23.
$$\begin{array}{rrrr} & x^3 & -3ax^2 & +2a^2x\\ & & x & +3a\\ \hline x^4 & -3ax^3 & +2a^2x^2 & \\ & +3ax^3 & -9a^2x^2 & +6a^3x\\ \hline x^4 & & -7a^2x^2 & +6a^3x\end{array}$$

24.
$$\begin{array}{rrrr} & a^3 & -4a^2b & +6ab^2\\ & & a^2 & +4ab\\ \hline a^5 & -4a^4b & +\ 6a^3b^2 & \\ & +4a^4b & -16a^3b^2 & +24a^2b^3\\ \hline a^5 & & -10a^3b^2 & +24a^2b^3\end{array}$$

25.
$$\begin{array}{rrrrrr} & & x^3 & -3x^2 & +2x & +1\\ & & & x^2 & +3x & +2\\ \hline x^5 & -3x^4 & +2x^3 & +\ x^2 & & \\ & +3x^4 & -9x^3 & +6x^2 & +3x & \\ & & +2x^3 & -6x^2 & +4x & +2\\ \hline x^5 & & -5x^3 & +\ x^2 & +7x & +2\end{array}$$

26.
$$\begin{array}{rrrrrr} & & x^3 & +x^2 & -2x & +1\\ & & & x^2 & -x & +2\\ \hline x^5 & +x^4 & -2x^3 & +\ x^2 & & \\ & -x^4 & -\ x^3 & +2x^2 & -\ x & \\ & & 2x^3 & +2x^2 & -4x & +2\\ \hline x^5 & & -\ x^3 & +5x^2 & -5x & +2\end{array}$$

27.

$$\begin{array}{rrrrr}
x^2 & +xy & +y^2 & & \\
x^2 & -xy & +y^2 & & \\
\hline
x^4 & +x^3y & +x^2y^2 & & \\
 & -x^3y & -x^2y^2 & -xy^3 & \\
 & & +x^2y^2 & +xy^3 & +y^4 \\
\hline
x^4 & & +x^2y^2 & & +y^4
\end{array}$$

28.

$$\begin{array}{rrrrr}
a^4 & +a^2b^2 & +b^4 & & \\
a^4 & -a^2b^2 & +b^4 & & \\
\hline
a^8 & +a^6b^2 & +a^4b^4 & & \\
 & -a^6b^2 & -a^4b^4 & -a^2b^6 & \\
 & & +a^4b^4 & +a^2b^6 & +b^8 \\
\hline
a^8 & & +a^4b^4 & & +b^8
\end{array}$$

29.

$$\begin{array}{rrrrrr}
2x^3 & -3x^2y & +2xy^2 & +y^3 & & \\
x^2 & +3xy & +2y^2 & & & \\
\hline
2x^5 & -3x^4y & +2x^3y^2 & +x^2y^3 & & \\
 & +6x^4y & -9x^3y^2 & +6x^2y^3 & +3xy^4 & \\
 & & +4x^3y^2 & -6x^2y^3 & +4xy^4 & +2y^5 \\
\hline
2x^5 & +3x^4y & -3x^3y^2 & +x^2y^3 & +7xy^4 & +2y^5
\end{array}$$

30.

$$\begin{array}{rrrrrr}
x^3 & -4x^2y & +6xy^2 & -3y^3 & & \\
3x^2 & -4xy & +5y^2 & & & \\
\hline
3x^5 & -12x^4y & +18x^3y^2 & -9x^2y^3 & & \\
 & -4x^4y & +16x^3y^2 & -24x^2y^3 & +12xy^4 & \\
 & & +5x^3y^2 & -20x^2y^3 & +30xy^4 & -15y^5 \\
\hline
3x^5 & -16x^4y & +39x^3y^2 & -53x^2y^3 & +42xy^4 & -15y^5
\end{array}$$

XI. PAGE 39.

21. $x^6+x^4+x^2+1+2x^3 \,.\, x^2+2x^3 \,.\, x+2x^3+2x^2 \,.\, x+2x^2+2x=$ &c.

22. $x^6+x^4+x^2+1+2x^3(-x^2)+2x^3 \,.\, x+2x^3(-1)+2(-x^2)\,x+2(-x^2)(-1)$
$+2x(-1)=$ &c.

23. $4x^6+x^4y^2+x^2y^4+9y^6+2(2x^3)(-x^2y)+2 \,.\, 2x^3 \,.\, xy^2+2(2x^3)(-3y^3)$
$+2(-x^2y)(xy^2)+2(-x^2y)(-3y^3)+2xy^2(-3y^3)=$ &c.

24. $4x^6+x^4y^2+4x^2y^4+y^6+2 \,.\, 2x^3(-x^2y)+2 \,.\, 2x^3 \,.\, 2xy^2+2 \,.\, 2x^3(-y^3)$
$+2(-x^2y) \,.\, 2xy^2+2(-x^2y)(-y^3)+2 \,.\, 2xy^2(-y^3)=$ &c.

25. $\{(x-y)+z\}\{(x-y)-z\}=(x-y)^2-z^2=x^2-2xy+y^2-z^2.$

26. $\{(x-2y)+4z\}\{(x-2y)-4z\}=(x-2y)^2-16z^2=x^2-4xy+4y^2-16z^2.$

27. $\{(3x-5z)-y\}\{(3x-5z)+y\}=(3x-5z)^2-y^2=9x^2-30xz+25z^2-y^2.$

28. $\{-x+(2y-3z)\}\{x+(2y-3z)\}=-x^2+(2y-3z)^2=$ &c.

29. $(x^2+y^2)^2-x^2y^2=x^4+x^2y^2+y^4.$

30. $(3x^2+2y^2)^2-x^2y^2=9x^4+11x^2y^2+4y^4.$ **31.** $(x^2-x)^2-7^2=$ &c.

32. $(2x^2+7)^2-(3x)^2=4x^4+19x^2+49.$

33. $(a+b)^2-(c+d)^2.$ **34.** $(2a-3b)^2-(2c-4d)^2.$

35. $(2a-2d)^2-(3b+c)^2.$ **36.** $(a-4c)^2-(3b-d)^2.$

XII. PAGE 41.

1.
$$\begin{array}{rrrrr}
3a^2 & +ab & -b^2 & & \\
a^2 & -2ab & -3b^2 & & \\
\hline
3a^4 & +a^3b & -a^2b^2 & & \\
 & -6a^3b & -2a^2b^2 & +2ab^3 & \\
 & & -9a^2b^2 & -3ab^3 & +3b^4 \\
\hline
3a^4 & -5a^3b & -12a^2b^2 & -ab^3 & +3b^4
\end{array}$$

2.
$$\begin{array}{rrrrr}
x^2 & -xy & -3y^2 & & \\
3x^2 & +xy & -y^2 & & \\
\hline
3x^4 & -3x^3y & -9x^2y^2 & & \\
 & +x^3y & -x^2y^2 & -3xy^3 & \\
 & & -x^2y^2 & +xy^3 & +3y^4 \\
\hline
3x^4 & -2x^3y & -11x^2y^2 & -2xy^3 & +3y^4
\end{array}$$

3.
$$\begin{array}{rrrrr}
5x^3 & -4x^2y & +3xy^2 & & \\
x^2 & +4xy & +6y^2 & & \\
\hline
5x^5 & -4x^4y & +3x^3y^2 & & \\
 & +20x^4y & -16x^3y^2 & +12x^2y^3 & \\
 & & +30x^3y^2 & -24x^2y^3 & +18xy^4 \\
\hline
5x^5 & +16x^4y & +17x^3y^2 & -12x^2y^3 & +18xy^4
\end{array}$$

4.
$$\begin{array}{rrrrr}
x^3 & -7x^2y & +3xy^2 & & \\
x^4 & +7x^3y & -3x^2y^2 & & \\
\hline
x^7 & -7x^6y & +3x^5y^2 & & \\
 & +7x^6y & -49x^5y^2 & +21x^4y^3 & \\
 & & -3x^5y^2 & +21x^4y^3 & -9x^3y^4 \\
\hline
x^7 & & -49x^5y^2 & +42x^4y^3 & -9x^3y^4
\end{array}$$

5.
$$\begin{array}{rrrrrrr}
3x^3 & -7x^2 & +5x & -3 & & & \\
2x^3 & +7x^2 & -5x & +4 & & & \\
\hline
6x^6 & -14x^5 & +10x^4 & -6x^3 & & & \\
 & 21x^5 & -49x^4 & +35x^3 & -21x^2 & & \\
 & & -15x^4 & +35x^3 & -25x^2 & +15x & \\
 & & & +12x^3 & -28x^2 & +20x & -12 \\
\hline
6x^6 & +7x^5 & -54x^4 & +76x^3 & -74x^2 & +35x & -12
\end{array}$$

6.
$$\begin{array}{rrrrrrr}
a^3 & +3a^2 & -7a & +6 & & & \\
3a^3 & -a^2 & +5a & -4 & & & \\
\hline
3a^6 & +9a^5 & -21a^4 & +18a^3 & & & \\
 & -a^5 & -3a^4 & +7a^3 & -6a^2 & & \\
 & & +5a^4 & +15a^3 & -35a^2 & +30a & \\
 & & & -4a^3 & -12a^2 & +28a & -24 \\
\hline
3a^6 & +8a^5 & -19a^4 & +36a^3 & -53a^2 & +58a & -24
\end{array}$$

7.
$$\begin{array}{rrrrr}
\frac{3}{2}x^2 & -\frac{5}{4}xy & +y^2 & & \\
\frac{1}{2}x^2 & -\frac{1}{4}xy & -y^2 & & \\
\hline
\frac{3}{4}x^4 & -\frac{5}{8}x^3y & +\frac{1}{2}x^2y^2 & & \\
 & -\frac{3}{8}x^3y & +\frac{5}{16}x^2y^2 & -\frac{1}{4}xy^3 & \\
 & & -\frac{3}{2}x^2y^2 & +\frac{5}{4}xy^3 & -y^4 \\
\hline
\frac{3}{4}x^4 & -x^3y & -\frac{11}{16}x^2y^2 & +xy^3 & -y^4
\end{array}$$

8.

$$\begin{array}{rrrrr}
\frac{1}{4}x^2 & -\frac{2}{3}xy & +12y^2 & & \\
12x^2 & +\frac{2}{3}xy & -\frac{1}{4}y^2 & & \\
\hline
3x^4 & -8x^3y & +144x^2y^2 & & \\
 & +\frac{1}{6}x^3y & -\frac{4}{9}x^2y^2 & +8xy^3 & \\
 & & -\frac{1}{16}x^2y^2 & +\frac{1}{6}xy^3 & -3y^4 \\
\hline
3x^4 & -7\frac{5}{6}x^3y & +143\frac{71}{144}x^2y^2 & +8\frac{1}{6}xy^3 & -3y^4
\end{array}$$

9.

$$\begin{array}{rrrrrrrrrr}
a^2 & +b^2 & +c^2 & -bc & -ca & -ab & & & & \\
a & +b & +c & & & & & & & \\
\hline
a^3 & +ab^2 & +ac^2 & -abc & -a^2c & -a^2b & & & & \\
 & -ab^2 & & -abc & & +a^2b & +b^3 & +bc^2 & -b^2c & \\
 & & -ac^2 & -abc & +a^2c & & & -bc^2 & +b^2c & +c^3 \\
\hline
a^3 & & & -3abc & & & +b^3 & & & +c^3
\end{array}$$

10.

$$\begin{array}{rrrrrrrrrr}
4a^2 & +9b^2 & +c^2 & -3bc & -2ca & -6ab & & & & \\
2a & +3b & +c & & & & & & & \\
\hline
8a^3 & +18ab^2 & +2ac^2 & -6abc & -4a^2c & -12a^2b & & & & \\
 & -18ab^2 & & -6abc & & +12a^2b & +27b^3 & +3bc^2 & -9b^2c & \\
 & & -2ac^2 & -6abc & +4a^2c & & & -3bc^2 & +9b^2c & +c^3 \\
\hline
8a^3 & & & -18abc & & & +27b^3 & & & +c^3
\end{array}$$

11.

$$\begin{array}{rrrrrrrrrr}
a^2 & +b^2 & +c^2 & +bc & +ca & -ab & & & & \\
a & +b & -c & & & & & & & \\
\hline
a^3 & +ab^2 & +ac^2 & +abc & +a^2c & -a^2b & & & & \\
 & -ab^2 & & +abc & & +a^2b & +b^3 & +bc^2 & +b^2c & \\
 & & -ac^2 & +abc & -a^2c & & & -bc^2 & -b^2c & -c^3 \\
\hline
a^3 & & & +3abc & & & +b^3 & & & -c^3
\end{array}$$

12.

$$\begin{array}{rrrrrrrrrr}
9a^2 & +b^2 & +9c^2 & +3bc & -9ca & +3ab & & & & \\
3a & -b & +3c & & & & & & & \\
\hline
27a^3 & +3ab^2 & +27ac^2 & +9abc & -27a^2c & +9a^2b & & & & \\
 & -3ab^2 & & +9abc & & -9a^2b & -b^3 & -9bc^2 & -3b^2c & \\
 & & -27ac^2 & +9abc & +27a^2c & & & +9bc^2 & +3b^2c & +27c^3 \\
\hline
27a^3 & & & +27abc & & & -b^3 & & & +27c^3
\end{array}$$

13. Here, and whenever it may be necessary, arrange *both* expressions in the same way. [Art. 51].

$$\begin{array}{rrrrr}
x^2 & +xy & +y^2 & & \\
x^2 & -xy & +y^2 & & \\
\hline
x^4 & +x^3y & +x^2y^2 & & \\
 & -x^3y & -x^2y^2 & -xy^3 & \\
 & & +x^2y^2 & +xy^3 & +y^4 \\
\hline
x^4 & & +x^2y^2 & & +y^4
\end{array}$$

14.

$$\begin{array}{rrrrr}
a^2 & -2ab & +4b^2 & & \\
a^2 & +2ab & +4b^2 & & \\
\hline
a^4 & -2a^3b & +4a^2b^2 & & \\
 & +2a^3b & -4a^2b^2 & +8ab^3 & \\
 & & +4a^2b^2 & -8ab^3 & +16b^4 \\
\hline
a^4 & & +4a^2b^2 & & +16b^4
\end{array}$$

15.
$$\begin{array}{rrrrr}
a^3x^3 + a^2x^2 + & ax & & & \\
a^2x^2 - ax & +1 & & & \\
\hline
a^5x^5 + a^4x^4 + & a^3x^3 & & & \\
- a^4x^4 - & a^3x^3 & - a^2x^2 & & \\
+ & a^3x^3 & + a^2x^2 & + ax & \\
\hline
a^5x^5 \quad + & a^3x^3 & & + ax &
\end{array}$$

N.B. The factors of a product may be taken in *any order*. [See Art. 39.]

16. The product $=(x-1)(x+1)(x^2+1)=(x^2-1)(x^2+1)=x^4-1$.

17. The product $=(a-x)(a+x)(a^2+x^2)=(a^2-x^2)(a^2+x^2)=a^4-x^4$.

18. The product $=(x-2y)(x+2y)(x^2+4y^2)=(x^2-4y^2)(x^2+4y^2)=x^4-16y^4$.

19. The product $=(3x-5y)(3x+5y)(9x^2+25y^2)=(9x^2-25y^2)(9x^2+25y^2)$
$=81x^4-625y^4$.

20. The product $=(x-y)(x+y)(x^2+y^2)(x^4+y^4)=(x^2-y^2)(x^2+y^2)(x^4+y^4)$
$=(x^4-y^4)(x^4+y^4)=x^8-y^8$.

21. The product $=\{(a-b)(a+b)(a^2+b^2)\}^2=\{(a^2-b^2)(a^2+b^2)\}^2$
$=(a^4-b^4)^2=a^8-2a^4b^4+b^8$.

22. The product $=\{(x^2-x+1)(x^2+x+1)\}^2=(x^4+x^2+1)^2$
$=x^8+x^4+1+2x^6+2x^4+2x^2$.

23. $(x^2-xy+y^2)(x^2+xy+y^2)=x^4+x^2y^2+y^4$;
and similarly $(x^4+x^2y^2+y^4)(x^4-x^2y^2+y^4)=x^8+x^4y^4+y^8$.

24. $(a^2+ab+b^2)(a^2-ab+b^2)(a^4-a^2b^2+b^4)=(a^4+a^2b^2+b^4)(a^4-a^2b^2+b^4)$
$=a^8+a^4b^4+b^8$.

25. $(x^2-ax+a^2)(x^2+ax+a^2)(a-x)(a+x)=(x^4+a^2x^2+a^4)(a^2-x^2)=a^6-x^6$.

26. $(x+y+z)(-x+y+z)(x-y+z)(x+y-z)=\{-x^2+(y+z)^2\}\{x^2-(y-z)^2\}$
$=(2yz+y^2+z^2-x^2)(2yz-y^2-z^2+x^2)$
$=4y^2z^2-(y^2+z^2-x^2)^2=2y^2z^2+2z^2x^2+2x^2y^2-x^4-y^4-z^4$.

27. $(2a)^3+3(2a)^2(3b)+3(2a)(3b)^2+(3b)^3$, &c.

28. $(2a)^3+3(2a)^2(-3b)+3(2a)(-3b)^2+(-3b)^3$, &c.

29. $(3a)^3+3(3a)^2(-2b)+3(3a)(-2b)^2+(-2b)^3$, &c.

30. $\{a+(b+c)\}^3=a^3+3a^2(b+c)+3a(b+c)^2+(b+c)^3$, &c.
Or. See Art. 251.

31. $\{a-(b-c)\}^3=a^3-3a^2(b-c)+3a(b-c)^2-(b-c)^3$, &c.

32. $\{a-(b+c)\}^3=a^3-3a^2(b+c)+3a(b+c)^2-(b+c)^3$, &c.

35. The left-side $=(x^2+yz+zx+xy)+(y^2+yz+zx+xy)$

$$+(z^2+yz+zx+xy)-(x^2+y^2+z^2+2yz+2zx+2xy)$$
$$=(yz+zx+xy).$$

36. Perform the operations indicated, and all the terms will be found to vanish.

Or thus: The given expression on the left

$$=\{(b-c)^3-(b-c)(c-a)(a-b)\}+\text{two similar terms}$$
$$=(b-c)\{(b-c)^2-(c-a)(a-b)\}+\ldots\ldots\ldots\ldots\ldots\ldots$$
$$=(b-c)\{a^2+b^2+c^2-bc-ca-ab\}+\ldots\ldots\ldots\ldots\ldots$$
$$=\{a^2+b^2+c^2-bc-ca-ab\}\{(b-c)+(c-a)+(a-b)\}$$
$$=0.$$

Or thus. We know that identically

$$x^3+y^3+z^3-3xyz=(x+y+z)(x^2+y^2+z^2-yz-zx-xy).$$

Put $x=b-c,\ y=c-a,\ z=a-b$; then

$$(b-c)^3+(c-a)^3+(a-b)^3-3(b-c)(c-a)(a-b)$$
$$=\{(b-c)+(c-a)+(a-b)\}\{(b-c)^2+(c-a)^2+(a-b)^2$$
$$-(b-c)(c-a)-(c-a)(a-b)-(a-b)(b-c)\}$$
$$=0, \text{ since } (b-c)+(c-a)+(a-b)=0.$$

37. $(x^2+y^2+z^2+2yz+2zx+2xy)-(x^2+y^2+z^2+2yz-2zx-2xy)$

$$+(x^2+y^2+z^2-2yz+2zx-2xy)-(x^2+y^2+z^2-2yz-2zx+2xy)=8xz.$$

38. $x^3+y^3+z^3+x^2y+x^2z+y^2z+y^2x$

$$+z^2x+z^2y-(y^2z+yz^2)-(z^2x+zx^2)-(x^2y+xy^2)=x^3+y^3+z^3.$$

39. $(b+c)(c+a)(a+b)$ will be found to be

$$2abc+a^2b+a^2c+b^2c+b^2a+c^2a+c^2b.$$

Also $(a+b+c)(bc+ca+ab)$ will be found to be

$$3abc+a^2b+a^2c+b^2c+b^2a+c^2a+c^2b.$$

40. $(2y+5z)^3=(2y)^3+3(2y)^2(5z)+3(2y)(5z)^2+(5z)^3$

$$=8y^3+60y^2z+150yz^2+125z^3=8y^3+125z^3+30yz(2y+5z).$$

Hence $\qquad x^3=8y^3+125z^3+3yzx.$

41. $x^2+y^2+z^2+2yz+2zx+2xy=(x+y+z)^2$

$$=\{(b+c-2a)+(c+a-2b)+(a+b-2c)\}^2=0.$$

42. $(b+c)(c+a)(a+b)$ will be found to be

$$2abc+a^2b+a^2c+b^2c+b^2a+c^2a+c^2b.$$

Hence $a^3+b^3+c^3+3(b+c)(c+a)(a+b)$

$$=a^3+b^3+c^3+3a^2b+\text{similar terms}+6abc=(a+b+c)^3.$$

43. Each side will be found to be $2b^2c^2+2c^2a^2+2a^2b^2+4abc\,(a+b+c)$.

44. The left side is $\{(a+b)+c\}^3$, and the right side is $\{a+(b+c)\}^3$.

45. We have

$$\begin{aligned} &\{(ax^2-a^2x+a^3)+(bx^2-b^2x+b^3)\}\{(ax^2-a^2x+a^3)-(bx^2-b^2x+b^3)\} \\ &\quad=(ax^2-a^2x+a^3)^2-(bx^2-b^2x+b^3)^2 \\ &\quad=(a^2-b^2)\,x^4-2\,(a^3-b^3)\,x^3+3\,(a^4-b^4)\,x^2-2\,(a^5-b^5)\,x+a^6-b^6. \end{aligned}$$

XIV. PAGE 50.

1.
$$\begin{array}{rl} x-2\,) & x^2-5x+6\,(\,x-3 \\ & \underline{x^2-2x\qquad\quad} \\ & \qquad -3x+6 \\ & \qquad -3x+6 \end{array}$$

2.
$$\begin{array}{rl} x+8\,) & x^2+5x-24\,(\,x-3 \\ & \underline{x^2+8x\qquad\qquad} \\ & \qquad -3x-24 \\ & \qquad -3x-24 \end{array}$$

3.
$$\begin{array}{rl} x-11\,) & x^2-23x+132\,(\,x-12 \\ & \underline{x^2-11x\qquad\qquad} \\ & \qquad -12x+132 \\ & \qquad -12x+132 \end{array}$$

4.
$$\begin{array}{rl} x+7\,) & x^2-\ 4x-77\,(\,x-11 \\ & \underline{x^2+\ 7x\qquad\qquad} \\ & \qquad -11x-77 \\ & \qquad -11x-77 \end{array}$$

5.
$$\begin{array}{rl} x-2\,) & 3x^2-4x-4\,(\,3x+2 \\ & \underline{3x^2-6x\qquad\quad} \\ & \qquad\quad 2x-4 \\ & \qquad\quad 2x-4 \end{array}$$

6.
$$\begin{array}{rl} 3x-5\,) & 3x^2-11x+10\,(\,x-2 \\ & \underline{3x^2-\ 5x\qquad\qquad} \\ & \qquad -\ 6x+10 \\ & \qquad -\ 6x+10 \end{array}$$

7.
$$\begin{array}{rl} a-b\,) & a^2-b^2\,(\,a+b \\ & \underline{a^2-ab\qquad} \\ & \qquad ab-b^2 \\ & \qquad ab-b^2 \end{array}$$

8.
$$\begin{array}{rl} x-3y\,) & x^2-9\ y^2\,(\,x+3y \\ & \underline{x^2-3xy\qquad\qquad} \\ & \qquad 3xy-9y^2 \\ & \qquad 3xy-9y^2 \end{array}$$

9.
$$\begin{array}{rl} x+4y\,) & x^2-16\ y^2\,(\,x-4y \\ & \underline{x^2+\ 4xy\qquad\qquad} \\ & \ -\ 4xy-16y^2 \\ & \ -\ 4xy-16y^2 \end{array}$$

10.
$$\begin{array}{rl} 3x+8y^2\,) & 9x^2-64\ y^4\,(\,3x-8y^2 \\ & \underline{9x^2+24xy^2\qquad\qquad} \\ & \qquad -24xy^2-64y^4 \\ & \qquad -24xy^2-64y^4 \end{array}$$

11.
$$\begin{array}{rl} x-\frac{1}{2}y^2\Big) & x^2-\frac{1}{4}\ y^4\Big(x+\frac{1}{2}y^2 \\ & \underline{x^2-\frac{1}{2}xy^2\qquad\qquad} \\ & \qquad \frac{1}{2}xy^2-\frac{1}{4}y^4 \\ & \qquad \frac{1}{2}xy^2-\frac{1}{4}y^4 \end{array}$$

12.
$$\begin{array}{rl} \frac{1}{3}x^3-\frac{1}{4}a^3\Big) & \frac{1}{9}x^6-\frac{1}{16}a^6\Big(\frac{1}{3}x^3+\frac{1}{4}a^3 \\ & \underline{\frac{1}{9}x^6-\frac{1}{12}a^3x^3\qquad\qquad} \\ & \qquad \frac{1}{12}a^3x^3-\frac{1}{16}a^6 \\ & \qquad \frac{1}{12}a^3x^3-\frac{1}{16}a^6 \end{array}$$

13.
$$\begin{array}{l} 6x+12\,\Big)\,3x^2+8x+4\,\Big(\,\frac{1}{2}x+\frac{1}{3} \\ \qquad\quad 3x^2+6x \\ \hline \qquad\qquad\quad 2x+4 \\ \qquad\qquad\quad 2x+4 \end{array}$$

14.
$$\begin{array}{l} 4x+3\,\Big)\,x^2-\frac{7}{12}x-1\,\Big(\,\frac{1}{4}x-\frac{1}{3} \\ \qquad\quad x^2+\frac{3}{4}x \\ \hline \qquad\qquad -\frac{4}{3}x-1 \\ \qquad\qquad -\frac{4}{3}x-1 \end{array}$$

15.
$$\begin{array}{l} a-b\,)\,a^3-b^3\,\,(\,a^2+ab+b^2 \\ \qquad\quad a^3-a^2b \\ \hline \qquad\qquad a^2b-b^3 \\ \qquad\qquad a^2b-ab^2 \\ \hline \qquad\qquad\qquad ab^2-b^3 \\ \qquad\qquad\qquad ab^2-b^3 \end{array}$$

16.
$$\begin{array}{l} 2a+3b\,)\,8a^3+27b^3\,\,(\,4a^2-6ab+9b^2 \\ \qquad\qquad 8a^3+12a^2b \\ \hline \qquad\qquad\quad -12a^2b+27b^3 \\ \qquad\qquad\quad -12a^2b-18ab^2 \\ \hline \qquad\qquad\qquad\qquad 18ab^2+27b^3 \\ \qquad\qquad\qquad\qquad 18ab^2+27b^3 \end{array}$$

17.
$$\begin{array}{l} ax+by\,)\,a^3x^3+\ b^3y^3\,\,(\,a^2x^2-abxy+b^2y^2 \\ \qquad\qquad a^3x^3+a^2bx^2y \\ \hline \qquad\qquad\quad -a^2bx^2y+b^3y^3 \\ \qquad\qquad\quad -a^2bx^2y-ab^2xy^2 \\ \hline \qquad\qquad\qquad\qquad ab^2xy^2+b^3y^3 \\ \qquad\qquad\qquad\qquad ab^2xy^2+b^3y^3 \end{array}$$

18.
$$\begin{array}{l} 2a^2x^2-5y^2\,)\,8a^6x^6-125y^6\,\,(\,4a^4x^4+10a^2x^2y^2+25y^4 \\ \qquad\qquad\qquad 8a^6x^6-20a^4x^4y^2 \\ \hline \qquad\qquad\qquad\qquad 20a^4x^4y^2-125y^6 \\ \qquad\qquad\qquad\qquad 20a^4x^4y^2-\ 50a^2x^2y^4 \\ \hline \qquad\qquad\qquad\qquad\qquad 50a^2x^2y^4-125y^6 \\ \qquad\qquad\qquad\qquad\qquad 50a^2x^2y^4-125y^6 \end{array}$$

19.
$$\begin{array}{l} -x+3\,)\,x^3-8x\ -3\,\,(\,-x^2-3x-1 \\ \qquad\quad x^3-3x^2 \\ \hline \qquad\qquad 3x^2-8x-3 \\ \qquad\qquad 3x^2-9x \\ \hline \qquad\qquad\qquad x-3 \\ \qquad\qquad\qquad x-3 \end{array}$$

N.B. In this example we are careful to arrange both dividend and divisor in the same way. [See Art. 67.]

20.
$$\begin{array}{l} -x+2\,)\,2x^3-5x^2+4\,\,(\,-2x^2+x+2 \\ \qquad\qquad 2x^3-4x^2 \\ \hline \qquad\qquad\quad -x^2+4 \\ \qquad\qquad\quad -x^2+2x \\ \hline \qquad\qquad\qquad\quad -2x+4 \\ \qquad\qquad\qquad\quad -2x+4 \end{array}$$

21.
$$\begin{array}{l} -x+3\,)\,2x^4-9x^3+10x^2-5x+6\,\,(\,-2x^3+3x^2-x+2 \\ \qquad\qquad 2x^4-6x^3 \\ \hline \qquad\qquad\quad -3x^3+10x^2-5x+6 \\ \qquad\qquad\quad -3x^3+\ 9x^2 \\ \hline \qquad\qquad\qquad\qquad x^2-5x+6 \\ \qquad\qquad\qquad\qquad x^2-3x \\ \hline \qquad\qquad\qquad\qquad\quad -2x+6 \\ \qquad\qquad\qquad\qquad\quad -2x+6 \end{array}$$

22.
$$\begin{array}{rrl} x^2-x+1\ ) & x^4+2x^2-x+2 & (\ x^2+x+2 \\ & \underline{x^4-x^3+x^2} & \\ & x^3+x^2-x+2 & \\ & \underline{x^3-x^2+x} & \\ & 2x^2-2x+2 & \\ & 2x^2-2x+2 & \end{array}$$

XV. Pages 53—54.

1.
$$\begin{array}{rrl} x^2+x+1\ ) & x^4+x^2+1 & (\ x^2-x+1 \\ & \underline{x^4+x^3+x^2} & \\ & -x^3+1 & \\ & \underline{-x^3-x^2-x} & \\ & x^2+x+1 & \\ & x^2+x+1 & \end{array}$$

2.
$$\begin{array}{rrl} 1-x^2+x^4\ ) & 1+x^4+x^8 & (\ 1+x^2+x^4 \\ & \underline{1-x^2+x^4} & \\ & x^2+x^8 & \\ & \underline{x^2-x^4+x^6} & \\ & x^4-x^6+x^8 & \\ & x^4-x^6+x^8 & \end{array}$$

3.
$$\begin{array}{rrl} x^2+2x+4\ ) & x^4+4x^2+16 & (\ x^2-2x+4 \\ & \underline{x^4+2x^3+4x^2} & \\ & -2x^3+16 & \\ & \underline{-2x^3-4x^2-8x} & \\ & 4x^2+8x+16 & \\ & 4x^2+8x+16 & \end{array}$$

4.
$$\begin{array}{rrl} 4x^2-6x+9\ ) & 16x^4+36x^2+81 & (\ 4x^2+6x+9 \\ & \underline{16x^4-24x^3+36x^2} & \\ & 24x^3+81 & \\ & \underline{24x^3-36x^2+54x} & \\ & 36x^2-54x+81 & \\ & 36x^2-54x+81 & \end{array}$$

5.
$$\begin{array}{rrl} x^2+x+1\ ) & x^4+2x^2+x+2 & (\ x^2-x+2 \\ & \underline{x^4+x^3+x^2} & \\ & -x^3+x^2+x+2 & \\ & \underline{-x^3-x^2-x} & \\ & 2x^2+2x+2 & \\ & 2x^2+2x+2 & \end{array}$$

6.
$$\begin{array}{rrl} x^2-x+1\ ) & 2x^4-x^3+2x^2+1 & (\ 2x^2+x+1 \\ & \underline{2x^4-2x^3+2x^2} & \\ & x^3+1 & \\ & \underline{x^3-x^2+x} & \\ & x^2-x+1 & \\ & x^2-x+1 & \end{array}$$

7.

$$\begin{array}{r}
-x+2\ )\ x^5-4x^3+2x-4\ (\ -x^4-2x^3-2 \\
x^5-2x^4 \\ \hline
2x^4-4x^3+2x-4 \\
2x^4-4x^3 \\ \hline
2x-4 \\
2x-4
\end{array}$$

8.

$$\begin{array}{r}
-x+2\ )\ x^6-3x^2-52\ (\ -x^5-2x^4-4x^3-8x^2-13x-26 \\
x^6-2x^5 \\ \hline
2x^5-3x^2-52 \\
2x^5-4x^4 \\ \hline
4x^4-3x^2-52 \\
4x^4-8x^3 \\ \hline
8x^3-3x^2-52 \\
8x^3-16x^2 \\ \hline
13x^2-52 \\
13x^2-26x \\ \hline
26x-52 \\
26x-52
\end{array}$$

9.

$$\begin{array}{r}
1-2x+3x^2\ )\ 1-x+5x^3-3x^4\ (\ 1+x-x^2 \\
1-2x+3x^2 \\ \hline
x-3x^2+5x^3 \\
x-2x^2+3x^3 \\ \hline
-x^2+2x^3-3x^4 \\
-x^2+2x^3-3x^4
\end{array}$$

10.

$$\begin{array}{r}
x^2+2x-1\ )\ x^6+2x^5-4x^4-2x^3+12x^2-2x-1\ (\ x^4-3x^2+4x+1 \\
x^6+2x^5-x^4 \\ \hline
-3x^4-2x^3+12x^2 \\
-3x^4-6x^3+3x^2 \\ \hline
4x^3+9x^2-2x \\
4x^3+8x^2-4x \\ \hline
x^2+2x-1 \\
x^2+2x-1
\end{array}$$

11.

$$\begin{array}{r}
1-2x+x^2\ )\ 1-4x^3+3x^4\ (\ 1+2x+3x^2 \\
1-2x+x^2 \\ \hline
2x-x^2-4x^3 \\
2x-4x^2+2x^3 \\ \hline
3x^2-6x^3+3x^4 \\
3x^2-6x^3+3x^4
\end{array}$$

12.

$$\begin{array}{r}
1-2x+x^2\ )\ 1-5x^4+4x^5\ (\ 1+2x+3x^2+4x^3 \\
1-2x+x^2 \\ \hline
2x-x^2 \\
2x-4x^2+2x^3 \\ \hline
3x^2-2x^3-5x^4 \\
3x^2-6x^3+3x^4 \\ \hline
4x^3-8x^4+4x^5 \\
4x^3-8x^4+4x^5
\end{array}$$

13.

$$\begin{array}{l}
7+5x-3x^2\;)\;35+4x-16x^2+19x^3-6x^4\;(\;5-3x+2x^2\\
\qquad\qquad\quad 35+25x-15x^2\\ \hline
\qquad\qquad\qquad -21x-x^2\\
\qquad\qquad\qquad -21x-15x^2+9x^3\\ \hline
\qquad\qquad\qquad\qquad 14x^2+10x^3-6x^4\\
\qquad\qquad\qquad\qquad 14x^2+10x^3-6x^4
\end{array}$$

14.

$$\begin{array}{l}
-x^3+3x^2-3x+1\;)\;3x^5-12x^4+17x^3-9x^2+1\;(\;-3x^2+3x+1\\
\qquad\qquad\qquad\quad 3x^5-9x^4+9x^3-3x^2\\ \hline
\qquad\qquad\qquad\qquad -3x^4+8x^3-6x^2+1\\
\qquad\qquad\qquad\qquad -3x^4+9x^3-9x^2+3x\\ \hline
\qquad\qquad\qquad\qquad\qquad -x^3+3x^2-3x+1\\
\qquad\qquad\qquad\qquad\qquad -x^3+3x^2-3x+1
\end{array}$$

15.

$$\begin{array}{l}
x^2-2x+1\;)\;x^5-5x^2+5x-1\;(\;x^3+2x^2+3x-1\\
\qquad\qquad\quad x^5-2x^4+x^3\\ \hline
\qquad\qquad\qquad 2x^4-x^3-5x^2+5x-1\\
\qquad\qquad\qquad 2x^4-4x^3+2x^2\\ \hline
\qquad\qquad\qquad\quad 3x^3-7x^2+5x-1\\
\qquad\qquad\qquad\quad 3x^3-6x^2+3x\\ \hline
\qquad\qquad\qquad\qquad -x^2+2x-1\\
\qquad\qquad\qquad\qquad -x^2+2x-1
\end{array}$$

16.

$$\begin{array}{l}
4x^2+11x+1\;)\;16x^6-97x^4-84x^2+77x+8\;(\;4x^4-11x^3+5x^2-11x+8\\
\qquad\qquad\quad 16x^6+44x^5+4x^4\\ \hline
\qquad\qquad\qquad -44x^5-101x^4-84x^2+77x+8\\
\qquad\qquad\qquad -44x^5-121x^4-11x^3\\ \hline
\qquad\qquad\qquad\qquad 20x^4+11x^3-84x^2+77x+8\\
\qquad\qquad\qquad\qquad 20x^4+55x^3+5x^2\\ \hline
\qquad\qquad\qquad\qquad\quad -44x^3-89x^2+77x+8\\
\qquad\qquad\qquad\qquad\quad -44x^3-121x^2-11x\\ \hline
\qquad\qquad\qquad\qquad\qquad\quad 32x^2+88x+8\\
\qquad\qquad\qquad\qquad\qquad\quad 32x^2+88x+8
\end{array}$$

17.

$$\begin{array}{l}
1+x+x^2\;)\;1+2x^5+x^6+2x^7\;(\;1-x+x^3-x^4+2x^5\\
\qquad\qquad\quad 1+x+x^2\\ \hline
\qquad\qquad\qquad -x-x^2\\
\qquad\qquad\qquad -x-x^2-x^3\\ \hline
\qquad\qquad\qquad\qquad x^3+2x^5\\
\qquad\qquad\qquad\qquad x^3+x^4+x^5\\ \hline
\qquad\qquad\qquad\qquad\quad -x^4+x^5+x^6\\
\qquad\qquad\qquad\qquad\quad -x^4-x^5-x^6\\ \hline
\qquad\qquad\qquad\qquad\qquad 2x^5+2x^6+2x^7\\
\qquad\qquad\qquad\qquad\qquad 2x^5+2x^6+2x^7
\end{array}$$

18.

$$\begin{array}{l}
x^2+x+1\;)\;x^{10}+x^5+1\;(\;x^8-x^7+x^5-x^4+x^3-x+1\\
\qquad\qquad\quad x^{10}+x^9+x^8\\
\hline
\qquad\qquad\qquad -x^9-x^8+x^5+1\\
\qquad\qquad\qquad -x^9-x^8-x^7\\
\hline
\qquad\qquad\qquad\qquad\qquad x^7+x^5+1\\
\qquad\qquad\qquad\qquad\qquad x^7+x^6+x^5\\
\hline
\qquad\qquad\qquad\qquad\qquad\quad -x^6+1\\
\qquad\qquad\qquad\qquad\qquad\quad -x^6-x^5-x^4\\
\hline
\qquad\qquad\qquad\qquad\qquad\qquad\qquad x^5+x^4+1\\
\qquad\qquad\qquad\qquad\qquad\qquad\qquad x^5+x^4+x^3\\
\hline
\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad -x^3+1\\
\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad -x^3-x^2-x\\
\hline
\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad x^2+x+1\\
\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad x^2+x+1
\end{array}$$

9.

$$\begin{array}{l}
2x^2-3xy+4y^2\;)\;6x^5-7x^4y+x^3y^2+20x^2y^3-22xy^4+8y^5\;(\;3x^3+x^2y-4xy^2+2y^3\\
\qquad\qquad\qquad\quad 6x^5-9x^4y+12x^3y^2\\
\hline
\qquad\qquad\qquad\qquad\quad 2x^4y-11x^3y^2+20x^2y^3-22xy^4+8y^5\\
\qquad\qquad\qquad\qquad\quad 2x^4y-3x^3y^2+4x^2y^3\\
\hline
\qquad\qquad\qquad\qquad\qquad\quad -8x^3y^2+16x^2y^3-22xy^4+8y^5\\
\qquad\qquad\qquad\qquad\qquad\quad -8x^3y^2+12x^2y^3-16xy^4\\
\hline
\qquad\qquad\qquad\qquad\qquad\qquad\qquad\quad 4x^2y^3-6xy^4+8y^5\\
\qquad\qquad\qquad\qquad\qquad\qquad\qquad\quad 4x^2y^3-6xy^4+8y^5
\end{array}$$

0.

$$\begin{array}{l}
4y^2-3xy+2x^2\;)\;8y^5-22xy^4+20x^2y^3+x^3y^2-7x^4y+6x^5\;(\;2y^3-4xy^2+x^2y+3x\\
\qquad\qquad\qquad\quad 8y^5-6xy^4+4x^2y^3\\
\hline
\qquad\qquad\qquad\qquad -16xy^4+16x^2y^3+x^3y^2-7x^4y+6x^5\\
\qquad\qquad\qquad\qquad -16xy^4+12x^2y^3-8x^3y^2\\
\hline
\qquad\qquad\qquad\qquad\qquad\qquad 4x^2y^3+9x^3y^2-7x^4y+6x^5\\
\qquad\qquad\qquad\qquad\qquad\qquad 4x^2y^3-3x^3y^2+2x^4y\\
\hline
\qquad\qquad\qquad\qquad\qquad\qquad\qquad\quad 12x^3y^2-9x^4y+6x^5\\
\qquad\qquad\qquad\qquad\qquad\qquad\qquad\quad 12x^3y^2-9x^4y+6x^5
\end{array}$$

21.

$$\begin{array}{l}
x-y\;)\;x^2-xz-y^2+yz\;(\;x+y-z\\
\qquad\quad x^2-xy\\
\hline
\qquad\qquad\quad xy-xz-y^2+yz\\
\qquad\qquad\quad xy\qquad\;\;-y^2\\
\hline
\qquad\qquad\qquad\;\; -xz\qquad+yz\\
\qquad\qquad\qquad\;\; -xz\qquad+yz
\end{array}$$

NOTE. In this and the following examples the arranging of the terms in the divisor and dividend, *and after each subtraction*, is of the utmost importance.

22.

$$\begin{array}{l}
x-y\;)\;x^2+2xz-y^2-2yz\;(\;x+y+2z\\
\qquad\quad x^2-xy\\
\hline
\qquad\qquad\quad xy+2xz-y^2-2yz\\
\qquad\qquad\quad xy\qquad\quad\;\;-y^2\\
\hline
\qquad\qquad\qquad +2xz\qquad-2yz\\
\qquad\qquad\qquad\;\;\; 2xz\qquad-2yz
\end{array}$$

23.

$$\begin{array}{l}
a+b+c\;)\;a^2+ac-b^2-bc\;(\;a-b\\
\qquad\qquad\;\; a^2+ab+ac\\
\hline
\qquad\qquad\qquad -ab-b^2-bc\\
\qquad\qquad\qquad -ab-b^2-bc
\end{array}$$

24.

$$\begin{array}{l}
-x+y-z\ )\ x^2-y^2+2yz-z^2\ (\ -x-y+z \\
\qquad x^2-xy+xz \\ \hline
\qquad\quad xy-xz-y^2+2yz-z^2 \\
\qquad\quad xy-y^2+yz \\ \hline
\qquad\qquad -xz+yz-z^2 \\
\qquad\qquad -xz+yz-z^2
\end{array}$$

25.

$$\begin{array}{l}
-a+2b+c\ )\ a^2-4b^2-4bc-c^2\ (\ -a-2b-c \\
\qquad a^2-2ab-ac \\ \hline
\qquad\quad 2ab+ac-4b^2-4bc-c^2 \\
\qquad\quad 2ab-4b^2-2bc \\ \hline
\qquad\qquad ac-2bc-c^2 \\
\qquad\qquad ac-2bc-c^2
\end{array}$$

26.

$$\begin{array}{l}
-a-2b+3c\ )\ a^2+4ab-6ac+4b^2-12bc+9c^2\ (\ -a-2b+3c \\
\qquad a^2+2ab-3ac \\ \hline
\qquad\quad 2ab-3ac+4b^2-12bc+9c^2 \\
\qquad\quad 2ab+4b^2-6bc \\ \hline
\qquad\qquad -3ac-6bc+9c^2 \\
\qquad\qquad -3ac-6bc+9c^2
\end{array}$$

27.

$$\begin{array}{l}
a+2b+2c\ )\ 2a^2+3ab+3ac-2b^2-4bc-2c^2\ (\ 2a-b-c \\
\qquad 2a^2+4ab+4ac \\ \hline
\qquad\quad -ab-ac-2b^2-4bc-2c^2 \\
\qquad\quad -ab-2b^2-2bc \\ \hline
\qquad\qquad -ac-2bc-2c^2 \\
\qquad\qquad -ac-2bc-2c^2
\end{array}$$

28.

$$\begin{array}{l}
a+4b-4c\ )\ 3a^2+8ab-11ac-16b^2+20bc-4c^2\ (\ 3a-4b+c \\
\qquad 3a^2+12ab-12ac \\ \hline
\qquad\quad -4ab+ac-16b^2+20bc-4c^2 \\
\qquad\quad -4ab-16b^2+16bc \\ \hline
\qquad\qquad ac+4bc-4c^2 \\
\qquad\qquad ac+4bc-4c^2
\end{array}$$

29.

$$\begin{array}{l}
x+y-z\ )\ x^3+3xyz+y^3-z^3\ (\ x^2-xy+xz+y^2+yz+z^2 \\
\qquad x^3+x^2y-x^2z \\ \hline
\qquad\quad -x^2y+x^2z+3xyz+y^3-z^3 \\
\qquad\quad -x^2y-xy^2+xyz \\ \hline
\qquad\qquad x^2z+xy^2+2xyz+y^3-z^3 \\
\qquad\qquad x^2z+xyz-xz^2 \\ \hline
\qquad\qquad\quad xy^2+xyz+xz^2+y^3-z^3 \\
\qquad\qquad\quad xy^2+y^3-y^2z \\ \hline
\qquad\qquad\qquad xyz+xz^2+y^2z-z^3 \\
\qquad\qquad\qquad xyz+y^2z-yz^2 \\ \hline
\qquad\qquad\qquad\quad xz^2+yz^2-z^3 \\
\qquad\qquad\qquad\quad xz^2+yz^2-z^3
\end{array}$$

Or thus: $x+y-z\;)\;x^3+3xyz+y^3-z^3\;(\;x^2-x(y-z)+y^2+yz+z^2$

$x^3+x^2(y-z)$

$-x^2(y-z)+3xyz$

$-x^2(y-z)-x(y-z)^2$

$x(y^2+yz+z^2)+y^3-z^3$

$x(y^2+yz+z^2)+(y-z)(y^2+yz+z^2)$

30. $-2x+y-z\;)\;8x^3+6xyz-y^3+z^3\;(\;-4x^2-2xy+2xz-y^2-yz-z^2$

$8x^3-4x^2y+4x^2z$

$4x^2y-4x^2z+6xyz-y^3+z^3$

$4x^2y-2xy^2+2xyz$

$-4x^2z+2xy^2+4xyz-y^3+z^3$

$-4x^2z\quad+2xyz-2xz^2$

$2xy^2+2xyz+2xz^2-y^3+z^3$

$2xy^2\quad-y^3+y^2z$

$2xyz+2xz^2-y^2z+z^3$

$2xyz\quad-y^2z+yz^2$

$2xz^2-yz^2+z^3$

$2xz^2-yz^2+z^3$

31. $3a-2b+c\;)\;27a^3+18abc-8b^3+c^3\;(\;9a^2+6ab-3ac+4b^2+2bc+c^2$

$27a^3-18a^2b+\;9a^2c$

$18a^2b-\;9a^2c+18abc-8b^3+c^3$

$18a^2b-12ab^2+\;6abc$

$-\;9a^2c+12ab^2+12abc-8b^3+c^3$

$-\;9a^2c\quad+\;6abc-3ac^2$

$12ab^2+\;6abc+3ac^2-8b^3+c^3$

$12ab^2\quad-8b^3+4b^2c$

$6abc+3ac^2-4b^2c+c^3$

$6abc\quad-4b^2c+2bc^2$

$3ac^2\quad-2bc^2+c^3$

$3ac^2\quad-2bc^2+c^3$

32. $a+b+c\;)\;a^3+3a^2b+3ab^2+b^3+c^3\;(\;a^2+2ab-ac+b^2-bc+c^2$

$a^3+\;a^2b+\;a^2c$

$2a^2b-\;a^2c+3ab^2+\quad b^3+\;c^3$

$2a^2b\quad+2ab^2+2abc$

$-\;a^2c+\;ab^2-2abc+\;b^3+c^3$

$-\;a^2c\quad-\;abc-ac^2$

$ab^2-\;abc+ac^2+b^3+c^3$

$ab^2\quad+b^3+b^2c$

$-\;abc+ac^2\quad-b^2c+c^3$

$-\;abc\quad-b^2c-bc^2$

$ac^2\quad+bc^2+c^3$

$ac^2\quad+bc^2+c^3$

Or thus: $(a+b)+c\;)\;(a+b)^3+\quad c^3\;(\;(a+b)^2-(a+b)c+c^2$

$(a+b)^3+(a+b)^2c$

$-(a+b)^2c+c^3$

$-(a+b)^2c-(a+b)c^2$

$(a+b)c^2+c^3$

$(a+b)c^2+c^3$

33.

$$\begin{array}{l}
a+b+c\ )\ a^4-2a^2b^2-2a^2c^2+b^4-2b^2c^2+c^4\ (\ a^3-a^2b-a^2c-ab^2+2abc-ac^2+b^3-b^2c-bc^2+c^3 \\
\qquad a^4+a^3b+a^3c \\
\hline
\qquad -a^3b-a^3c-2a^2b^2-2a^2c^2+b^4-2b^2c^2+c^4 \\
\qquad -a^3b\qquad -a^2b^2-a^2bc \\
\hline
\qquad\quad -a^3c-a^2b^2+a^2bc-2a^2c^2+b^4-2b^2c^2+c^4 \\
\qquad\quad -a^3c\qquad -a^2bc-a^2c^2 \\
\hline
\qquad\qquad -a^2b^2+2a^2bc-a^2c^2+b^4-2b^2c^2+c^4 \\
\qquad\qquad -a^2b^2-ab^3-ab^2c \\
\hline
\qquad\qquad\quad 2a^2bc-a^2c^2+ab^3+ab^2c+b^4-2b^2c^2+c^4 \\
\qquad\qquad\quad 2a^2bc\qquad +2ab^2c+2abc^2 \\
\hline
\qquad\qquad\qquad -a^2c^2+ab^3-ab^2c-2abc^2+b^4-2b^2c^2+c^4 \\
\qquad\qquad\qquad -a^2c^2\qquad -abc^2-ac^3 \\
\hline
\qquad\qquad\qquad\quad ab^3-ab^2c-abc^2+ac^3+b^4-2b^2c^2+c^4 \\
\qquad\qquad\qquad\quad ab^3\qquad +b^4+b^3c \\
\hline
\qquad\qquad\qquad\qquad -ab^2c-abc^2+ac^3-b^3c-2b^2c^2+c^4 \\
\qquad\qquad\qquad\qquad -ab^2c\qquad -b^3c-b^2c^2 \\
\hline
\qquad\qquad\qquad\qquad\quad -abc^2+ac^3-b^2c^2+c^4 \\
\qquad\qquad\qquad\qquad\quad -abc^2\qquad -b^2c^2-bc^3 \\
\hline
\qquad\qquad\qquad\qquad\qquad ac^3+bc^3+c^4 \\
\qquad\qquad\qquad\qquad\qquad ac^3+bc^3+c^4
\end{array}$$

34. $\{(x+1)^3+y^3\}\div(x+y+1)=\{(x+1)^3+y^3\}\div\{(x+1)+y\}=(x+1)^2-(x+1)\,y+y^2$, by Art. 69, formula 5.

35. $\{(x^2-yz)^3+8y^3z^3\}\div(x^2+yz)=\{(x^2-yz)^3+(2yz)^3\}\div\{(x^2-yz)+2yz\}$

$$=(x^2-yz)^2-(x^2-yz)\,(2yz)+(2yz)^2=x^4-4x^2yz+7y^2z^2.$$

36. $\{(x^2-2yz)^3-27y^3z^3\}\div(x^2-5yz)=\{(x^2-2yz)^3-(3yz)^3\}\div\{(x^2-2yz)-(3yz)\}$

$$=(x^2-2yz)^2+(x^2-2yz)\,(3yz)+(3yz)^2 \qquad \text{[Art. 69]}$$

$$=x^4-x^2yz+7y^2z^2.$$

37. $(9a^2+6ab+b^2-4c^2-4cd-d^2)\div(3a+b-2c-d)=\{(3a+b)^2-(2c+d)^2\}\div\{(3a+b)-(2c+d)\}=(3a+b)+(2c+d).$

38.
$$
\begin{array}{l}
x-(a-2b)\,\big)\,x^3+(4ab-b^2)\,x-(a-2b)\,(a^2+3b^2)\,\big(\,x^2+(a-2b)\,x+(a^2+3b^2)\\
\qquad\qquad\quad \underline{x^3-(a-2b)\,x^2\qquad\qquad\qquad\qquad\qquad\quad}\\
\qquad\qquad\qquad (a-2b)\,x^2+(4ab-b^2)\,x-(a-2b)\,(a^2+3b^2)\\
\qquad\qquad\qquad \underline{(a-2b)\,x^2-(a^2-4ab+4b^2)\,x\qquad\qquad\qquad\quad}\\
\qquad\qquad\qquad\qquad\qquad (a^2+3b^2)\,x-(a-2b)\,(a^2+3b^2)\\
\qquad\qquad\qquad\qquad\qquad (a^2+3b^2)\,x-(a-2b)\,(a^2+3b^2)
\end{array}
$$

39. First arrange both expressions according to descending powers of a.

$$
\begin{array}{l}
a^2(b-c)-a\,(b^2-c^2)+b^2c-bc^2\,\big)\,a^3(b-c)-a\,\,(b^3-c^3)+b^3c-bc^3\,\big(\,a+(b+c)\\
\qquad\qquad\qquad\qquad\qquad\quad \underline{a^3(b-c)-a^2(b^2-c^2)+a\,(b^2c-bc^2)\qquad\qquad\qquad}\\
\qquad\qquad\qquad\qquad\qquad\qquad\quad a^2(b^2-c^2)-a\,(b^3+b^2c-bc^2-c^3)+b^3c-bc^3\\
\qquad\qquad\qquad\qquad\qquad\qquad\quad a^2(b^2-c^2)-a\,(b^3+b^2c-bc^2-c^3)+b^3c-bc^3
\end{array}
$$

40.
$$
\begin{array}{l}
a^2(b-c)-a\,(b^2-c^2)+b^2c-bc^2\,\big)\,a^4(b-c)-a\,\,(b^4-c^4)+b^4c-bc^4\,\big(\,a^2+a\,(b+c)+(b^2+bc+c^2)\\
\qquad\qquad\qquad\qquad\qquad\quad \underline{a^4(b-c)-a^3(b^2-c^2)+a^2(b^2c-bc^2)\qquad\qquad\qquad}\\
\qquad\qquad\qquad\qquad\qquad\qquad\quad a^3(b^2-c^2)-a^2(b^2c-bc^2)-a\,(b^4-c^4)+b^4c-bc^4\\
\qquad\qquad\qquad\qquad\qquad\qquad\quad \underline{a^3(b^2-c^2)-a^2(b^3+b^2c-bc^2-c^3)+a\,(b^3c-bc^3)\qquad}\\
\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad +a^2(b^3-c^3)-a\,(b^4+b^3c-bc^3-c^4)+b^4c-bc^4\\
\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\;\; a^2(b^3-c^3)-a\,(b^4+b^3c-bc^3-c^4)+b^4c-bc^4
\end{array}
$$

MISCELLANEOUS EXAMPLES. I. PAGES 54—56.

A. 1. $(0-1)^2+(1+1)^2+(-1-0)^2=1+4+1=6.$

2.

$$\begin{array}{rrrrrr}
3a^2 & -2ab & -2ac & & & \\
 & 3ab & & +2b^2 & +3bc & \\
 & & -2ac & & -2bc & +c^2 \\
\hline
3a^2 & +ab & -4ac & +2b^2 & +bc & +c^2
\end{array}$$

3. $a^3+b^3+c^3-3abc-a^3+abc-b^3+abc-c^3+abc=0.$

4.

$$\begin{array}{rrrr}
 & 2x^2 & -\frac{1}{2}x & +\frac{1}{3} \\
 & & \frac{1}{2}x & +3 \\
\hline
x^3 & -\frac{1}{4}x^2 & +\frac{1}{6}x & \\
 & 6x^2 & -\frac{3}{2}x & +1 \\
\hline
x^3 & +\frac{23}{4}x^2 & -\frac{4}{3}x & +1
\end{array}
\qquad
\begin{array}{rrrrr}
\frac{1}{4}x^2 & +\frac{1}{2}xy & +y^2 & & \\
\frac{1}{4}x^2 & -\frac{1}{2}xy & +y^2 & & \\
\hline
\frac{1}{16}x^4 & +\frac{1}{8}x^3y & +\frac{1}{4}x^2y^2 & & \\
 & -\frac{1}{8}x^3y & -\frac{1}{4}x^2y^2 & -\frac{1}{2}xy^3 & \\
 & & \frac{1}{4}x^2y^2 & +\frac{1}{2}xy^3 & +y^4 \\
\hline
\frac{1}{16}x^4 & & +\frac{1}{4}x^2y^2 & & +y^4
\end{array}$$

Or thus:— $\left(\frac{1}{4}x^2+\frac{1}{2}xy+y^2\right)\left(\frac{1}{4}x^2-\frac{1}{2}xy+y^2\right)$

$$=\left\{\left(\frac{1}{4}x^2+y^2\right)+\frac{1}{2}xy\right\}\left\{\left(\frac{1}{4}x^2+y^2\right)-\frac{1}{2}xy\right\}=\left(\frac{1}{4}x^2+y^2\right)^2-\frac{1}{4}x^2y^2$$

$$=\frac{1}{16}x^4+\frac{1}{4}x^2y^2+y^4.$$

5. $(x+1)(x+2)(x+3)$ will be found to be $x^3+6x^2+11x+6$, and $(x-1)(x-2)(x-3)$ will be found to be $x^3-6x^2+11x-6$.

Hence given expression

$$=x^3+6x^2+11x+6-(x^3-6x^2+11x-6)=12x^2+12.$$

6.

$$1-2x+x^2\;)\;2-12x^5+10x^6\;(\;2+4x+6x^2+8x^3+10x^4$$

$$\begin{array}{rrrrrrr}
2 & -4x & +2x^2 & & & & \\
\hline
 & 4x & -2x^2 & & & & \\
 & 4x & -8x^2 & +4x^3 & & & \\
\hline
 & & 6x^2 & -4x^3 & & & \\
 & & 6x^2 & -12x^3 & +6x^4 & & \\
\hline
 & & & 8x^3 & -6x^4 & -12x^5 & \\
 & & & 8x^3 & -16x^4 & +8x^5 & \\
\hline
 & & & & 10x^4 & -20x^5 & +10x^6 \\
 & & & & 10x^4 & -20x^5 & +10x^6
\end{array}$$

B. 1.

$$6^2+3\,(6+11)=36+51=87;$$
$$\sqrt{2\times 3\times 6}-11=\ 6-11=-5.$$

2.

$$\begin{array}{lr}
5b^4-3ab^3+4a^2b^2 & \\
\qquad\qquad 4a^2b^2 & -\,3a^3b+5a^4 \\
\hline
5b^4-3ab^3 & +\,3a^3b-5a^4
\end{array}$$

3.

$$x-(1-1+x)=x-1+1-x=0,$$
$$3x-\overline{7-4x}=3x-7+4x=7x-7,$$
$$\begin{aligned}b-2a-\{c-a-(b-a+c)\}&=b-2a-\{c-a-b+a-c\}\\&=b-2a-c+a+b-a+c=2b-2a.\end{aligned}$$

4.

$$\begin{array}{ll}
9x^2-1 & \\
\ \ x^2+\dfrac{1}{9} & \\
\hline
9x^4-\ x^2 & \\
\qquad\quad x^2 & -\dfrac{1}{9} \\
\hline
9x^4 & -\dfrac{1}{9}
\end{array}$$

$$\{(a+b)+c\}\,\{(a+b)-c\}=(a+b)^2-c^2=a^2+2ab+b^2-c^2.$$

5. $2\,(a^2+b^2+c^2-bc-ca-ab)=2a^2+2b^2+2c^2-2bc-2ca-2ab,$

also
$$\begin{aligned}(b-c)^2+(c-a)^2+(a-b)^2&=\ b^2-2bc+c^2+c^2-2ca+a^2+a^2-2ab+b^2\\&=2a^2+2b^2+2c^2-2bc-2ca-2ab.\end{aligned}$$

6.

$$\begin{array}{lll}
x+y-1\ ) & x^3-3x^2+3x+y^3-1 & (\ x^2-x\,(y+2)+y^2+y+1 \\
& \underline{x^3+\ x^2\,(y-1)\qquad\qquad} & \\
& \quad -\ x^2\,(y+2)+3x & \\
& \quad \underline{-\ x^2\,(y+2)-\ x\,(y-1)\,(y+2)\qquad\quad} & \\
& \qquad\qquad\qquad x\,(y^2+y+1)+y^3-1 & \\
& \qquad\qquad\qquad x\,(y^2+y+1)+(y-1)\,(y^2+y+1) &
\end{array}$$

Or thus: $(x^3-3x^2+3x+y^3-1)\div(x+y-1)$

$$=\{(x-1)^3+y^3\}\div\{(x-1)+y\}=(x-1)^2-(x-1)\,y+y^2.$$

C. 1.

$$\frac{\sqrt{4^2+2\times3\times(-2)}}{4}+\frac{\sqrt{3^2+(-2)\times4}}{3}+\frac{\sqrt{(-2)^2+4\times3}}{-2}$$
$$=\frac{\sqrt{16-12}}{4}+\frac{\sqrt{9-8}}{3}+\frac{\sqrt{4+12}}{-2}=\frac{2}{4}+\frac{1}{3}+\frac{4}{-2}=-\frac{7}{6}.$$

3. $\{(ax^2+a)-x\}\times\{(ax^2+a)+x\}=(ax^2+a)^2-x^2=a^2x^4+2a^2x^2-x^2+a^2.$

$(2a+b-3c)^2=(2a)^2+b^2+(-3c)^2+2\,(2a)\,b+2\,(2a)\,(-3c)+2b\,(-3c)=$&c.

4. $x^2+2xy+y^2-(x^2-y^2)-\{2xy-x^2-2xy+y^2\}$

$$=x^2+2xy+y^2-x^2+y^2-2xy+x^2+2xy-y^2=x^2+2xy+y^2.$$

5. Expression required $=(x^6+x^5y+x^4y^2-x^3y^3+y^6)\div(x^2+xy+y^2)$

$$\begin{array}{l}
x^2+xy+y^2\ )\ x^6+x^5y+x^4y^2-x^3y^3+y^6\ (\ x^4-xy^3+y^4\\
\qquad\qquad\quad x^6+x^5y+x^4y^2\\
\hline
\qquad\qquad\qquad\qquad\quad -x^3y^3+y^6\\
\qquad\qquad\qquad\qquad\quad -x^3y^3-x^2y^4-xy^5\\
\hline
\qquad\qquad\qquad\qquad\qquad\qquad x^2y^4+xy^5+y^6\\
\qquad\qquad\qquad\qquad\qquad\qquad x^2y^4+xy^5+y^6
\end{array}$$

6. (i) $ab-ac+bc-ba+ca-cb=0$, (ii) $(b-c)(c-a)(a-b)$ will be found to be

$$bc^2-b^2c+ca^2-c^2a+ab^2-a^2b.$$

Hence we have

$$a^2b-a^2c+b^2c-b^2a+c^2a-c^2b+(bc^2-b^2c+ca^2-c^2a+ab^2-a^2b)=0,$$

since all the terms cancel.

D. 1.
$$6\times1+3\times2+4(-3)=6+6-12=0,$$
$$1^3+2^3+(-3)^3-3\times1\times2(-3)=1+8-27+18=0.$$

2. $3\{x-2y+2z\}-[4y+\{2y-z+x\}]=3x-6y+6z-[4y+2y-z+x]$
$$=3x-6y+6z-4y-2y+z-x=2x-12y+7z.$$

3. $(a-b-c)^2-(a+b+c)^2=a^2+b^2+c^2-2ab-2ac+2bc$
$$-(a^2+b^2+c^2+2ab+2ac+2bc)=-4ab-4ac.$$

4.
$$\begin{array}{l}
\qquad\qquad a^3+b^3\\
\qquad\qquad a\ -b\\
\hline
\qquad\qquad a^4+ab^3\\
\qquad\qquad\quad -a^3b-b^4\\
\hline
a+b\ )\ a^4-\ a^3b+ab^3-b^4\ (\ a^3-2a^2b+2ab^2-b^3\\
\qquad\qquad a^4+\ a^3b\\
\hline
\qquad\qquad\quad -2a^3b\\
\qquad\qquad\quad -2a^3b-2a^2b^2\\
\hline
\qquad\qquad\qquad\qquad 2a^2b^2+\ ab^3\\
\qquad\qquad\qquad\qquad 2a^2b^2+2ab^3\\
\hline
\qquad\qquad\qquad\qquad\qquad -\ ab^3-b^4\\
\qquad\qquad\qquad\qquad\qquad -\ ab^3-b^4
\end{array}$$

5.
$$\begin{array}{l}
x+z\ )\ x^3+z^3\ \ (\ x^2-xz+z^2\\
\qquad\quad x^3+x^2z\\
\hline
\qquad\qquad -x^2z+z^3\\
\qquad\qquad -x^2z-xz^2\\
\hline
\qquad\qquad\qquad\quad xz^2+z^3\\
\qquad\qquad\qquad\quad xz^2+z^3
\end{array}$$

Hence $\{(x+y)^3+z^3\}\div\{(x+y)+z\}=(x+y)^2-(x+y)z+z^2.$

6. (i) $x^2-y^2+y^2-z^2+z^2-x^2=0,$

(ii) $(x-y)^2+(y-z)^2+(z-x)^2=x^2-2xy+y^2+y^2-2yz+z^2+z^2-2zx+x^2$
$$=2x^2+2y^2+2z^2-2xy-2yz-2zx.$$

Again $2(x-y)(x-z)+2(y-z)(y-x)+2(z-x)(z-y)$

$$=2(x^2-xy-xz+yz)+2(y^2-xy+xz-yz)+2(z^2+xy-xz-yz)$$
$$=2x^2+2y^2+2z^2-2xy-2xz-2yz.$$

E. 1. $$\frac{2\sqrt{(3^2+4^2)}+\sqrt[3]{(3^2+4^2-5^2+6\,.\,6^2)}}{6-5+4-3}$$
$$=\frac{2\sqrt{25}+\sqrt[3]{(9+16-25+216)}}{2}=\frac{10+6}{2}=8.$$

2. $4x-y+x-3\{2y-3x-3y\}=4x-y+x-6y+9x+9y=14x+2y.$

$2x-3y-4x+8y+5\{3x-2x+2y\}=2x-3y-4x+8y+15x-10x+10y$
$$=3x+15y.$$

$$\begin{array}{r}
14x+2y \\
3x+15y \\
\hline
42x^2+6xy \\
210xy+30y^2 \\
\hline
42x^2+216xy+30y^2
\end{array}$$

3. $(a+b)^3=a^3+3a^2b+3ab^2+b^3=a^3+b^3+3(a^2b+ab^2)=a^3+b^3+3ab(a+b).$

$(1-2)^3=-1$, and $1+(-2)^3+3\,.\,1(-2)(1-2)=1-8+6=-1.$

4. Put down only the terms which contain x^2.

$$\begin{array}{l}
x^2-(a-b)x-ab \\
x^2+(a+b)x+ab \\
\hline
\qquad\qquad -abx^2 \\
\qquad\qquad -(a-b)(a+b)x^2 \\
\qquad\qquad +abx^2 \\
\qquad\qquad \overline{-(a^2-b^2)x^2}
\end{array}$$

The required coefficient is therefore $-(a^2-b^2)$.

5. $(2a+3b)^2-(3a+2b)^2=4a^2+12ab+9b^2-(9a^2+12ab+4b^2)$
$$=-5a^2+5b^2=-5(a^2-b^2).$$

And $(a^2-b^2)\div(a+b)=a-b.\quad\therefore\ -5(a^2-b^2)\div(a+b)=-5(a-b).$

$(2a-3b)^3+(3a-2b)^3=8a^3-36a^2b+54ab^2-27b^3+27a^3-54a^2b+36ab^2-8b^3$
$$=35a^3-90a^2b+90ab^2-35b^3.$$

$$\begin{array}{r|l}
a-b\ ) & 35a^3-90a^2b+90ab^2-35b^3\ (\ 35a^2-55ab+35b^2 \\
& \underline{35a^3-35a^2b} \\
& -55a^2b+90ab^2 \\
& \underline{-55a^2b+55ab^2} \\
& 35ab^2-35b^3 \\
& 35ab^2-35b^3
\end{array}$$

6. $\{(a+b)+(c-d)\}\{(a+b)-(c-d)\}+\{(a-b)+(c+d)\}\{-(a-b)+(c+d)\}$
$$=(a+b)^2-(c-d)^2+(c+d)^2-(a-b)^2$$
$$=a^2+2ab+b^2-c^2+2cd-d^2+c^2+2cd+d^2-a^2+2ab-b^2=4ab+4cd.$$

F. 1. $$\frac{(3+4)(5-4)-(4+5)(-4+3)}{3\,.\,4+4\,.\,5-5(-4)-(-4)\,3}=\frac{7+9}{12+20+20+12}=\frac{1}{4}.$$

2.
$$\begin{array}{l}
a-x\\
a-\frac{1}{2}y\\ \hline
a^2-ax\\
\quad-\frac{1}{2}ay+\frac{1}{2}xy\\ \hline
a^2-a\left(x+\frac{1}{2}y\right)+\frac{1}{2}xy\\
a-\frac{1}{4}z\\ \hline
a^3-a^2\left(x+\frac{1}{2}y\right)+\frac{1}{2}axy\\
\quad-\frac{1}{4}a^2z\qquad\qquad+\frac{1}{4}a\left(x+\frac{1}{2}y\right)z-\frac{1}{8}xyz\\ \hline
a^3-a^2\left(x+\frac{1}{2}y+\frac{1}{4}z\right)+a\left(\frac{1}{2}xy+\frac{1}{4}xz+\frac{1}{8}yz\right)-\frac{1}{8}xyz
\end{array}$$

3.
$$\begin{array}{rl}
x+4\ ) & x^2+7x+c\ (\ x+3\\
& x^2+4x\\ \cline{2-2}
& \qquad 3x+c\\
& \qquad 3x+12\\ \cline{2-2}
& \qquad\qquad c-12
\end{array}$$

If x^2+7x+c is *exactly* divisible by $x+4$, there will be no remainder; but the remainder will vanish only when $c-12=0$, that is, when $c=12$.

4.
$$\begin{array}{rl}
x+y-z\ ) & x^2-y^2+2yz-z^2\ (\ x-(y-z)\\
& x^2+x(y-z)\\ \cline{2-2}
& \quad -x(y-z)-y^2+2yz-z^2\\
& \quad -x(y-z)-(y-z)^2
\end{array}$$

$$\begin{array}{rl}
x-\frac{1}{2}y\ \Big) & 8x^4+\ xy^3-y^4\ \Big(\ 8x^3+4x^2y+2xy^2+\ldots\\
& 8x^4-4x^3y\\ \cline{2-2}
& \qquad 4x^3y+\ xy^3\\
& \qquad 4x^3y-2x^2y^2\\ \cline{2-2}
& \qquad\qquad 2x^2y^2+xy^3\\
& \qquad\qquad 2x^2y^2-xy^3\\ \cline{2-2}
& \qquad\qquad\quad \&c.
\end{array}$$

Thus the required coefficient is $2y^2$.

5. Each expression $=a^2c^2+b^2c^2+a^2d^2+b^2d^2$.

6. $$\begin{aligned}(b-c)^2+(a-b)(a-c)&=b^2+c^2-2bc+a^2-ab-ac+bc\\ &=a^2+b^2+c^2-bc-ca-ab,\end{aligned}$$

and similarly for the other expressions.

G. 1. $5x-3[2x+9y-2\{3x-4y+4x\}]=5x-3[2x+9y-6x+8y-8x]$

$=5x-6x-27y+18x-24y+24x=41x-51y.$

2. $2a^2-4ab+2b^2-(6a^2-4ab+7b^2)-(4a^2-6ab+5b^2)$

$=2a^2-4ab+2b^2-6a^2+4ab-7b^2-4a^2+6ab-5b^2$

$=2a^2-6a^2-4a^2-4ab+4ab+6ab+2b^2-7b^2-5b^2=-8a^2+6ab-10b^2.$

3.

$$\begin{array}{llllllllll}
a^2 & -ab & -a & +b^2 & -b & +1 & & & & \\
a & +b & +1 & & & & & & & \\
\hline
a^3 & -a^2b & -a^2 & +ab^2 & -ab & +a & & & & \\
 & +a^2b & & -ab^2 & -ab & & +b^3 & -b^2 & +b & \\
 & & +a^2 & & -ab & -a & & +b^2 & -b & +1 \\
\hline
a^3 & & & & -3ab & & +b^3 & & & +1
\end{array}$$

4. $\{a^3+b^3+3a^2b+3ab^2-8c^3\}\div\{(a+b)-2c\}$

$=\{(a+b)^3-(2c)^3\}\div\{(a+b)-2c\}=(a+b)^2+2c(a+b)+(2c)^2.$ [Art. 69.

5. The other expression $=(x^7-64x)\div(x^2-2x+4)$.

$$\begin{array}{r|l|l}
x^2-2x+4 & x^7-64x & x^5+2x^4-8x^2-16x \\
 & x^7-2x^6+4x^5 & \\
\hline
 & 2x^6-4x^5 & \\
 & 2x^6-4x^5+8x^4 & \\
\hline
 & -8x^4 & \\
 & -8x^4+16x^3-32x^2 & \\
\hline
 & -16x^3+32x^2-64x & \\
 & -16x^3+32x^2-64x &
\end{array}$$

6. $(1+x+x^2)(1-x+x^2)$ will be found to be $1+x^2+x^4$.

Change x into x^2; then $(1+x^2+x^4)(1-x^2+x^4)=1+x^4+x^8$.

Again change x into x^2; then $(1+x^4+x^8)(1-x^4+x^8)=1+x^8+x^{16}$.

Hence $(1+x+x^2)(1-x+x^2)(1-x^2+x^4)(1-x^4+x^8)$

$=(1+x^2+x^4)(1-x^2+x^4)(1-x^4+x^8)=(1+x^4+x^8)(1-x^4+x^8)=1+x^8+x^{16}.$

$a^3(b-c)+b^3(c-a)+c^3(a-b)=a^3b-a^3c+b^3c-b^3a+c^3a-c^3b.$

Also $(b-c)(c-a)(a-b)$ will be found to be $a^2c-a^2b+ab^2-ac^2-b^2c+bc^2$.

Multiply by $a+b+c$; thus

$$\begin{array}{llllllllll}
-a^2b & +a^2c & +ab^2 & -ac^2 & -b^2c & +bc^2 & & & & \\
a & +b & +c & & & & & & & \\
\hline
-a^3b & +a^3c & +a^2b^2 & -a^2c^2 & -ab^2c & +abc^2 & & & & \\
 & & -a^2b^2 & & & -abc^2 & +a^2bc & +ab^3 & -b^3c & +b^2c^2 \\
 & & & +a^2c^2 & +ab^2c & & -a^2bc & -ac^3 & & -b^2c^2+bc^3 \\
\hline
-a^3b & +a^3c & +ab^3 & -ac^3 & -b^3c & +bc^3 & & & &
\end{array}$$

The sum of the two expressions is now obviously zero.

XVI. Pages 62—63.

1. $3x-x=10-4$; *i.e.* $2x=6$; $\therefore\ x=3$.

2. $x-4x=4-7$; *i.e.* $-3x=-3$; $\therefore\ x=1$.

3. $5x-6x=12-8$; *i.e.* $-x=4$; $\therefore\ x=-4$.

4. $7x-5x=7-19$; *i.e.* $2x=-12$; $\therefore\ x=-6$.

5. Removing brackets, $3x-6=2x-6$; $\therefore\ 3x-2x=-6+6$; *i.e.* $x=0$.

6. Removing brackets, $5x+10=3x+9+1$; or $5x+10=3x+10$;
$\therefore\ 5x-3x=10-10$; *i.e.* $2x=0$; $\therefore\ x=0$.

7. Removing brackets, $x-4+2x=7x-7$; $\therefore\ x+2x-7x=4-7$;

$$i.e.\ -4x=-3;\ \therefore\ x=\frac{3}{4}.$$

8. Removing brackets, $20-15x=21-28x$; $\therefore\ 28x-15x=21-20$;

$$i.e.\ 13x=1;\ \therefore\ x=\frac{1}{13}.$$

9. Removing brackets, $2x-6=5x+5+2x-1$;
$\therefore\ 2x-5x-2x=5-1+6$; *i.e.* $-5x=10$; $\therefore\ x=-2$.

10. Removing brackets, $4-4x+6+3x=13$;
$\therefore\ -4x+3x=13-6-4$; *i.e.* $-x=3$; $\therefore\ x=-3$.

11. Removing brackets, $2x-4+3x-9+4x-16-20=0$;

$$\therefore\ 2x+3x+4x=4+9+16+20;\ i.e.\ 9x=49;\ \therefore\ x=\frac{49}{9}.$$

12. Removing brackets, $2x-2-3x+6+4x-12+2=0$;
$\therefore\ 2x-3x+4x=2-6+12-2$; *i.e.* $3x=6$; $\therefore\ x=2$.

13. Removing brackets, $5x+6x+6-7x-14-8x-24=0$;
$\therefore\ 5x+6x-7x-8x=-6+14+24$; *i.e.* $-4x=32$; $\therefore\ x=-8$.

14. Multiply by 3; then $3x+2x=30$; *i.e.* $5x=30$; $\therefore\ x=6$.

15. Multiply by 20; then $4x-5x=20$; *i.e.* $-x=20$; $\therefore\ x=-20$.

16. Multiply by 6; then $3(x-1)+2(x-2)=18$;
$\therefore\ 3x-3+2x-4=18$; $\therefore\ 3x+2x=18+3+4$; *i.e.* $5x=25$; $\therefore\ x=5$.

17. Multiply by 12; then $3(x+1)-8(x-1)=36$;
$\therefore\ 3x+3-8x+8=36$; $\therefore\ 3x-8x=36-3-8$; *i.e.* $-5x=25$; $\therefore\ x=-5$.

18. Multiply by 10; then $5(2-x)-2(5x+21)=10(x+3)$;

$$\therefore\ 10-5x-10x-42=10x+30;\ \text{whence } x=-\frac{62}{25}.$$

19. Multiply by 12; then $6(x-2)+4(x-3)+3(x-4)=120$;
$\therefore\ 6x-12+4x-12+3x-12=120$; whence $x=12$.

20. Multiply by 12; then $6(x+1)+4(x+2)+3(x+4)+96=0$;
$\therefore\ 6x+6+4x+8+3x+12+96=0$; whence $x=-\frac{122}{13}$.

21. Multiply by 6; then $3(x-5)-2(x-4)=3(x-3)-6(x-2)$;
$\therefore\ 3x-15-2x+8=3x-9-6x+12$; whence $x=\frac{5}{2}$.

22. Multiply by 12; then $6(x+1)+4(x+2)+3(x+3)=192$;
$\therefore\ 6x+6+4x+8+3x+9=192$; whence $x=13$.

23. Remove brackets; thus $2x-[3-\{4x+x-1\}-5]=8$;
or $2x-[3-4x-x+1-5]=8$; or $2x-3+4x+x-1+5=8$; whence $x=1$.

24. $1-2\{x-3-3x\}=0$; or $1-2x+6+6x=0$; whence $x=-\frac{7}{4}$.

25. Remove brackets; then $x^2+3x+2=x^2-6x+8$;
$\therefore\ x^2+3x-x^2+6x=8-2$; *i.e.* $9x=6$; $\therefore\ x=\frac{2}{3}$.

26. Remove brackets; then $x^2-3x+2=x^2-7x+12$;
$\therefore\ x^2-3x-x^2+7x=12-2$; *i.e.* $4x=10$; $\therefore\ x=\frac{5}{2}$.

27. Remove brackets; then $2x^2=x^2+2x+1+x^2+6x+9$;
whence, by transposing $-8x=10$; $\therefore\ x=-\frac{5}{4}$.

28. Remove brackets; then $3x^2=x^2+2x+1+x^2+4x+4+x^2+6x+9$;
whence by transposing $-12x=14$; *i.e.* $x=-\frac{7}{6}$.

29. Remove brackets; then $x^2-7x+10+x^2-7x+12=2x^2-18x+40$;
whence, by transposing $4x=18$; *i.e.* $x=\frac{9}{2}$.

30. We have $x^2-2x+1+4x^2-24x+36=5x^2+50x+125$;
whence, by transposing $-76x=88$; *i.e.* $x=-\frac{22}{19}$.

31. We have $5x^2+10x+5+7x^2+42x+63=12x^2+48x+48$;
whence $4x=-20$; *i.e.* $x=-5$.

32. We have $x^2-5x+4=2x+x^2-5x+6$; whence $-2x=2$; *i.e.* $x=-1$.

33. Removing the brackets in the equation

$x^3-3x^2+3x-1+x^3-6x^2+12x-8+x^3-9x^2+27x-27=3x^3-18x^2+33x-18;$

$\therefore\ x^3-3x^2+3x+x^3-6x^2+12x+x^3-9x^2+27x-3x^3+18x^2-33x$

$=-18+1+8+27;$ *i.e.* $9x=18;$ or $x=2.$

34. Multiply by 10; then $5\left(x+\frac{1}{2}\right)-2\left(2x-\frac{1}{2}\right)+\frac{25}{2}=0;$

multiply by 2; then $5(2x+1)-2(4x-1)+25=0;$

$\therefore\ 10x+5-8x+2+25=0;$ whence $x=-16.$

35. Multiply by 8; then $3x+5-4(21+x)=40x-120;$

$\therefore\ 3x+5-84-4x=40x-120;$ whence $-41x=-41;$ *i.e.* $x=1.$

36. Multiply by 100; then $50x+375=525x-100;$

$\therefore\ -475x=-475;$ *i.e.* $x=1.$

37. Multiply by 1000; then $250x+4000-375x=200x-9000;$

whence $-325x=-13000;\ \therefore\ x=40.$

38. Multiply by 10000; then $1500x+12000-8750x+3750=625x;$

whence $-7875x=-15750;\ \therefore\ x=2.$

39. Multiply by 100; then $120x-\frac{1}{5}(18x-5)=40x+890;$

multiply by 5; then $600x-(18x-5)=200x+4450;$

$\therefore\ 600x-18x+5=200x+4450;$

whence $382x=4445,$ *i.e.* $x=\frac{4445}{382}.$

40. We have $ax-a^2=bx-b^2;\ \therefore\ ax-bx=-b^2+a^2;$

i.e. $x(a-b)=a^2-b^2;\ \therefore\ x=\frac{a^2-b^2}{a-b}=a+b.$

41. We have $2x-2a+3x-6a=2a;\ \therefore\ 2x+3x=2a+2a+6a;$

whence $x=2a.$

42. Multiply by 6; then $3(x+a+b)+2(x+a-b)=6b;$

$\therefore\ 3x+3a+3b+2x+2a-2b=6b;$ whence $5x=5b-5a;$ *i.e.* $x=b-a.$

43. We have $[a+b+a-b]x=a^2,$ or $2ax=a^2;\ \therefore\ x=\frac{a}{2}.$

44. We have $x[a+b+b-a]=b^2;\ \therefore\ 2bx=b^2;\ x=\frac{b}{2}.$

45. Multiply by 12; then $6(a+x)+4(2a+x)+3(3a+x)=36a;$

$\therefore\ 6a+6x+8a+4x+9a+3x=36a;$ whence $13x=13a,$ *i.e.* $x=a.$

46. Multiply by ab; then $xa^2+xb^2=ab(a^2+b^2);$

$\therefore\ x=ab(a^2+b^2)\div(a^2+b^2)=ab.$

47. We have $ab+a^2x+b^2x+abx^2=abx^2-ab$;

whence $$a^2x+b^2x=-2ab;\ \therefore\ x=\frac{-2ab}{a^2+b^2}.$$

48. We have $a^2b^2+a^2x+b^2x+x^2=a^2b^2+2abx+x^2$;

$\therefore\ a^2x+b^2x+x^2-2abx-x^2=a^2b^2-a^2b^2;\ \therefore\ a^2x+b^2x-2abx=0;\ \therefore\ x=0.$

49. We have $ax+a^2+b^2-bx=2ab$, or $ax-bx=2ab-a^2-b^2$;

whence $$x=\frac{2ab-a^2-b^2}{a-b}=-(a-b).$$

50. We have

$$x^2+a^2+b^2+2ax+2bx+2ab+x^2+a^2+b^2+2ax-2bx-2ab=2x^2;$$

or $$x^2+2ax+2bx+x^2+2ax-2bx-2x^2=-a^2-b^2-2ab-a^2-b^2+2ab,$$

$$i.e.\ 4ax=-2a^2-2b^2;\ \therefore\ x=\frac{-2a^2-2b^2}{4a}=-\frac{a^2+b^2}{2a}.$$

51. We have $x^2-ax-bx+ab+a^2+2ab+b^2=x^2+ax+bx+ab$;

$\therefore\ -ax-bx+a^2+2ab+b^2=ax+bx;\ \therefore\ -ax-bx-ax-bx=-a^2-2ab-b^2;$

$$\therefore\ -2ax-2bx=-a^2-2ab-b^2;\ \therefore\ x=\frac{-a^2-2ab-b^2}{-2a-2b}=\frac{a+b}{2}.$$

52. We have $(\overline{x+a}+\overline{b+c})\,(\overline{x+a}-\overline{b+c})=(\overline{x-a}-\overline{b-c})\,(\overline{x-a}+\overline{b-c})$;

$$\therefore\ (x+a)^2-(b+c)^2=(x-a)^2-(b-c)^2;$$

$$\therefore\ x^2+2ax+a^2-b^2-2bc-c^2=x^2-2ax+a^2-b^2+2bc-c^2;$$

$$\therefore\ 2ax-2bc=-2ax+2bc;\ \text{whence } 4ax=4bc, \text{ and } x=bc/a.$$

53. We have $ax^2+a^2x+bx^2+b^2x=ax^2+a^2x+2abx+a^2b+bx^2+b^2x+ab^2$;

$$\therefore\ 0=2abx+a^2b+ab^2;\ i.e.\ -2abx=a^2b+ab^2;\ \therefore\ x=\frac{a^2b+ab^2}{-2ab}=-\frac{a+b}{2}.$$

54. We have

$$x^3-3x^2a+3xa^2-a^3+x^3-3x^2b+3xb^2-b^3+x^3-3x^2c+3xc^2-c^3$$
$$=3x^3-3ax^2-3bx^2-3cx^2+3abx+3acx+3bcx-3abc;$$

$$\therefore\ 3xa^2-a^3+3xb^2-b^3+3xc^2-c^3=3abx+3acx+3bcx-3abc;$$

$$\therefore\ 3xa^2+3xb^2+3xc^2-3abx-3acx-3bcx=-3abc+a^3+b^3+c^3;$$

$$i.e.\ x=\frac{a^3+b^3+c^3-3abc}{3(a^2+b^2+c^2-ab-ac-bc)}=\frac{a+b+c}{3}\ \text{(by § 70, Ex. 2).}$$

Or thus: $(x-a)^3+(x-b)^3+(x-c)^3-3(x-a)(x-b)(x-c)=0$;

$\therefore\ \{\overline{x-a}+\overline{x-b}+\overline{x-c}\}\{(\overline{x-a}-\overline{x-b})^2+(\overline{x-b}-\overline{x-c})^2+\&c.\}=0;$

[See Art. 70, Ex. 2.]

$i.e.\ (3x-a-b-c)\,\{(a-b)^2+(b-c)^2$

$$+(c-a)^2-(a-b)(b-c)-(b-c)(c-a)-(c-a)(a-b)\}=0;$$

$$\therefore\ 3x-a-b-c=0, \text{ or } x=\frac{1}{3}(a+b+c).$$

XVII. Pages 67—71.

1. Let x be the larger number, then the other will be $200-x$.
$\therefore\ x-(200-x)=2$, whence $x=101$, and $200-x=99$.

2. Let x be the larger number, then the other will be $56-x$.
$\therefore\ x-(56-x)=20$, whence $x=38$, and $56-x=18$.

3. Let x be one part, then the other will be $25-x$.
$\therefore\ x-(25-x)=5$, whence $x=15$, and $25-x=10$.

4. Let x be one part, then the other will be $100-x$.
$\therefore\ x-(100-x)=45$, whence $x=72\frac{1}{2}$ and $100-x=27\frac{1}{2}$.

5. Let x be the number. Then $x+40=3x$, whence $x=20$.

6. Let x be the number. Then $x-14=\frac{1}{3}x$, whence $x=21$.

7. Let x be the larger number then $x-15$ will be the smaller.
$\therefore\ x=4(x-15)$, whence $x=20$ and $x-15=5$.

8. Let x be the larger number, then the smaller will be $x-10$.
$\therefore\ x=3(x-10)$, whence $x=15$ and $x-10=5$.

9. Let x be the larger number, then the smaller will be $38-x$.
$\therefore\ x=2(38-x)+2$, whence $x=26$ and $38-x=12$.

10. Let x be the larger number, then the smaller will be $31-x$.
$\therefore\ 31-x=\frac{1}{2}x-2$, whence $x=22$ and $31-x=9$.

11. Let x be the number. Then $\frac{x}{4}-\frac{x}{5}=2$, whence $x=40$.

12. Let $x=$ the number. Then $\frac{x}{3}-\frac{x}{7}=80$, whence $x=420$.

13. Let $x=$ the number. Then $4x-35=35-x$, whence $x=14$.

14. Let $x=$ the number. Then $8x-27=27-x$, whence $x=6$.

15. Let $x=$ the number. Then $4\left(\frac{x}{4}-\frac{x}{5}\right)=\frac{x}{3}-\frac{x}{7}+4$, whence $x=420$.

16. Let $x=$ the number. Then $50\left(\frac{x}{7}-\frac{x}{8}\right)=\frac{x}{2}+44$, whence $x=112$.

17. Let x and $100-x$ be the parts. Then $3x+5(100-x)=410$, whence $x=45$.

18. Let x and $100-x$ be the parts. Then $2x=3(100-x)$, whence $x=60$ and $100-x=40$.

19. Let x be the larger number and $x-20$ the smaller. Then

$$\frac{1}{2}(x-20)=\frac{1}{5}x, \text{ whence } x=33\tfrac{1}{3} \text{ and } x-20=13\tfrac{1}{3}.$$

20. Let x be the greater and $36-x$ the smaller number.

$$\therefore\ x-(36-x)=\frac{x}{2}, \text{ whence } x=24 \text{ and } 36-x=12.$$

21. Let $x=$ the number of pounds A must give B; then A would have $100-x$ and B $20+x$.

$$\therefore\ 20+x=\frac{1}{2}(100-x), \text{ whence } x=20.$$

22. Let x be the number of shillings A has. If A lose he will have $x-5$ shillings, and this is what B will have *after winning* 5*s*. Hence B must have had $x-5-5$ shillings, that is $x-10$ shillings.

Again, if A win he will have $x+5$ shillings, and B will have $x-10-5$ shillings. But in this case A will have three times as much as B. Hence $x+5=3(x-15)$, whence $x=25$.

23. Let A have x pounds, then B must have $50-x$ pounds. If A win x pounds he will have $x+x$, and B will have $50-x-x$. Hence

$$2x=3(50-2x);\ \therefore\ x=18\tfrac{3}{4}.$$

Thus A has £18. 15*s*. and B has £50 − £18. 15*s*. = £31. 5*s*.

24. Let $x=$ the whole number of pounds. Then A has $\frac{x}{2}$, B has $\frac{x}{3}$ and C has 50.

Hence $\frac{x}{2}+\frac{x}{3}+50=x$, whence $x=300$.

Thus A has £150 and B has £100.

25. Let x be the number of pounds that A has. Then B must have $x-5$ pounds. Hence $x+(x-5)=75$, whence $x=40$.

26. Let x be the number of pounds that C has. Then B has $x+20$ and A has $x+25$. Hence $(x+25)+(x+20)+x=360$, whence $x=105$.

27. Let A have x pounds; then B must have $x-15$, and C must have

$$x-15+5=x-10.$$

Hence $x+(x-15)+(x-10)=65$, whence $x=30$.

28. Let A have x pounds. Then B must have $x+5$ pounds, and C must have $x+(x+5)=2x+5$ pounds.

Hence $x+(x+5)+(2x+5)=100$, whence $x=22\tfrac{1}{2}$.

29. Let each child have x shillings; then each woman will have $2x$ shillings, and each man will have $x+15$ shillings. Hence, the whole number of shillings being 600, we have

$$10(x+15)+2x\times 20+x\times 40=600,$$

whence $x=5$.

30. Let each man receive x shillings, then each woman will receive $x-1$ shillings, and each child $\frac{1}{2}(x-1)$ shillings. Hence $3x+5(x-1)+10(x-1)=15$, whence $x=1\frac{2}{3}$. Thus each man gets 1*s.* 8*d.*, each woman 8*d.* and each child 4*d.*

31. Let x be the number of years. Their ages will then be $40+x$ and $10+x$ respectively. Hence $40+x=3(10+x)$, whence $x=5$.

32. Suppose the first man was twice as old as the second x years ago. Their ages were then $70-x$ and $50-x$. Hence $70-x=2(50-x)$, whence $x=30$.

33. Suppose the father to be x years old; then the son must be $\frac{x}{3}$ years old. In 10 years their ages will be $x+10$ and $\frac{x}{3}+10$ respectively. Hence $x+10=2\left(\frac{x}{3}+10\right)$, whence $x=30$.

34. Let x be the father's age in years. Then in 5 years the father will be $x+5$; and the daughter will therefore be $\frac{1}{4}(x+5)$. After 5 years more the father will be $x+10$ and the daughter will be $\frac{1}{4}(x+5)+5$. Hence $x+10=3\left\{\frac{1}{4}(x+5)+5\right\}$, whence $x=35$.

35. Let x be *A*'s share in pounds. Then $60-x$ will be *B*'s share, and $65-x$ will be *C*'s share. But *B* and *C* have together 75. Hence

$$60-x+65-x=75, \text{ whence } x=25.$$

Thus the shares are £25, £35 and £40.

36. Let x be *A*'s share in pounds. Then *B* must have $49-x$, *C* must have $51-x$, and *D* must have $47-x$. But *B* and *C* have £53 between them. Hence $(49-x)+(51-x)=53$, whence $x=23\frac{1}{2}$, &c.

37. Let the price be x shillings a pound. Then, for 20 shillings, $\frac{20}{x}$ pounds could be bought. If the price be reduced by 20 per cent., it would be $\frac{80}{100}x=\frac{4x}{5}$, and the number of pounds which could be bought for 20*s.* would be $20\div\frac{4x}{5}=\frac{25}{x}$. Hence $\frac{25}{x}=\frac{20}{x}+6$, whence $x=\frac{5}{6}$. The price is therefore $\frac{5}{6}s.=10d.$

38. Let the price be x pence per score. Then, for £1=240*d.*, $\frac{240}{x}$ score could be bought. If the price be raised 20 per cent., the price would be $\frac{120}{100}x=\frac{6}{5}x$; and the number of score which could be bought for 240*d.* would be $240\div\frac{6}{5}x=\frac{200}{x}$. Hence $\frac{240}{x}=\frac{200}{x}+4$, whence $x=10$.

39. Let £x be the man's whole property. Then his wife had £$\frac{x}{2}$, his son had £$\frac{x}{3}$, and his daughter £2000. Hence $x = \frac{x}{2} + \frac{x}{3} + 2000$, whence $x = 12000$.

40. Let the share of the youngest child be £x, then the second will have £$2x$ and the eldest £$4x$. Hence $4x - x = 750$, whence $x = 250$. The shares are therefore £1000, £500 and £250.

41. Let x be the number of shillings, then there are $\frac{x}{5}$ half-sovereigns, and therefore $28 - x - \frac{x}{5}$ sovereigns. The total value in shillings is therefore $x + 10 \times \frac{x}{5} + 20\left(28 - x - \frac{x}{5}\right)$. Thus $x + 2x + 20\left(28 - x - \frac{x}{5}\right) = 140$, whence $x = 20$.

42. Let x be the number of sovereigns, then there must be $3x$ half-sovereigns, and $36 - x - 3x$ shillings. The total value in shillings is therefore $20x + 30x + 36 - x - 3x$. Hence $20x + 30x + 36 - x - 3x = 220$, whence $x = 4$.

43. Let the required time be x minutes after five. At 5 o'clock the hour-hand is 25 minute spaces in front of the minute-hand, and at x minutes past 5 they are together, hence the hour-hand must have moved through $x - 25$ minute spaces in x minutes. But the minute-hand moves 12 times as fast as the hour-hand, and therefore $x = 12(x - 25)$, whence $x = \frac{300}{11} = 27\frac{3}{11}$.

44. Let the hands be at right angles at x minutes past 9. At 9 o'clock the hands are at right angles and the minute-hand is 45 minute spaces behind the hour-hand; also when the hands are next at right angles the minute-hand must be 15 minute spaces behind the hour-hand. Hence while the minute-hand describes x minute spaces the hour-hand will have described $x - 30$ minute spaces, and one hand moves 12 times as fast as the other. Thus $x = 12(x - 30)$, whence $x = 32\frac{8}{11}$.

45. Let x be the larger number, then the smaller will be $x - 3$. Hence $x^2 = (x - 3)^2 + 99$, whence $x = 18$.

46. Let x be the whole number of gallons. Then the number of gallons of spirit and of water will be $\frac{x}{2} + 25$ and $\frac{x}{3} - 5$ respectively. Hence

$$\left(\frac{x}{2} + 25\right) + \left(\frac{x}{3} - 5\right) = x, \text{ whence } x = 120.$$

47. Let x be the number in the reinforcement. When the reinforcement joined, the provisions would have lasted 1000 men for 50 days, or 1000×50 men for 1 day. But we are told that they lasted $(1000 + x)$ men for 20 days, there was therefore just enough for $(1000 + x) \times 20$ men for 1 day. Hence $1000 \times 50 = (1000 + x) \times 20$, whence $x = 1500$.

48. Let x be the number of days on which he worked, then he was idle for $36 - x$ days. Hence the number of shillings he received was

$$2\tfrac{1}{2} \times x - 1\tfrac{1}{2} \times (36 - x).$$

Hence $$\frac{5}{2}x - \frac{3}{2}(36 - x) = 58, \text{ whence } x = 28.$$

49. Let the whole sum divided be x pounds. Then the first receives $\frac{x}{3}+10$, and there was left $x-\left(\frac{x}{3}+10\right)=\frac{2}{3}x-10$.

The second then received $\frac{1}{2}\left(\frac{2}{3}x-10\right)+15$, and there was left

$$\frac{2}{3}x-10-\left\{\frac{1}{2}\left(\frac{2}{3}x-10\right)+15\right\}=\frac{1}{3}x-20.$$

Hence $\frac{1}{3}x-20=70$, whence $x=270$.

50. Let x be the number of sovereigns. Then the first had $\frac{x}{2}+1$, and $\frac{x}{2}-1$ were left. The second had $\frac{1}{2}\left(\frac{x}{2}-1\right)+1$, and $\frac{1}{2}\left(\frac{x}{2}-1\right)-1$ were left. Hence $\frac{1}{2}\left(\frac{x}{2}-1\right)-1=6$, whence $x=30$.

51. Let £x be the sum of money. After $\frac{x}{2}+20$ has been taken away, $\frac{x}{2}-20$ will be left. After $\frac{1}{3}\left(\frac{x}{2}-20\right)+30$ has been taken from $\frac{x}{2}-20$, there will be $\frac{2}{3}\left(\frac{x}{2}-20\right)-30$ left. Now when 40 more than a fifth part of this last remainder is taken, nothing is left; ∴ 40 must be $\frac{4}{5}$ths of this last remainder, which must therefore be 50. Hence $\frac{2}{3}\left(\frac{x}{2}-20\right)-30=50$, whence $x=280$.

52. Let x shillings be the charge for each dinner. The cost of the whole was $24x\times\frac{100}{112\frac{1}{2}}=\frac{64}{3}x$. The receipt was $21x$, and this was 1 less than the cost. Hence $21x+1=\frac{64}{3}x$, whence $x=3$.

53. Suppose A worked for x days, then B worked for $25-x$ days. Now A does $\frac{1}{30}$ th of the whole each day and B does $\frac{1}{20}$ th each day. Hence

$$\frac{1}{30}x+\frac{1}{20}(25-x)=1, \text{ whence } x=15.$$

54. Let x be the number of days for which A worked; then B worked for $x+10$ days. Now A does $\frac{1}{20}$ of the whole each day and B does $\frac{1}{30}$ of the whole each day. Hence $\frac{1}{20}x+\frac{1}{30}(x+10)=1$, whence $x=8$.

55. Let x be the number of men in each outer side of the larger square. Then, since this square is 3 deep, there would be $x-6$ men in each side of the square which would just fit into this. Hence the total number of the men in this hollow square is the difference of the squares of x and $x-6$, that is $x^2-(x-6)^2$.

The number of men in the side of the smaller of the two hollow squares is $x-6$ since it would just fit into the larger. Hence, as this square is 5 deep, there would be $x-6-10$ men in each side of the square which would just fit into this. Hence the total number of the men in this smaller hollow square is the difference between the square of $x-6$ and the square of $x-16$, that is $(x-6)^2-(x-16)^2$.

By the question $x^2-(x-6)^2=(x-6)^2-(x-16)^2$, whence $x=23$. The number in each hollow square is therefore 240.

XVIII. Page 76.

1. Subtract; then $2x=2$; $\therefore x=1$. Substitute in first equation; then

$$7+4y=1;\quad \therefore y=-\frac{3}{2}.$$

2. Multiply by 5 and 3 respectively; then $15x+25y=95$, $15x-12y=21$. By subtraction, $37y=74$ or $y=2$. Then $3x+10=19$; $\therefore x=3$.

3. Multiply 1st equation by 9; then, by addition $(111-99)y=9+99$; $\therefore y=9$. Then $x-99=1$; $\therefore x=100$.

4. Multiply by 3 and 4 respectively; then

$$24x-63y=15,\quad 24x+56y=-104.$$

Hence, by subtraction $-(63+56)y=15+104$; whence $y=-1$. Then

$$8x+21=5, \text{ whence } x=-2.$$

5. The L.C.M. of 34 and 51 is 102. Multiply by 3 and 2 respectively. Then we have $102x-45y=12$, $102x+50y=202$. By subtraction

$$-95y=-190;\quad \therefore y=2, \text{ \&c.}$$

6. The L.C.M. of 39 and 65 is 195, and multiplying by 5 and 3 respectively, we have $195x-75y=465$, $195x+51y=339$. By subtraction

$$126y=-126, \text{ or } y=-1, \text{ \&c.}$$

7. We see that $85=17\times 5$ and $119=17\times 7$, hence we multiply by 7 and 5 respectively. Then $133x+595y=2450$, $85x+595y=2210$. By subtraction $48x=240$; $\therefore x=5$, &c.

8. Multiply by 25 and 8 respectively; then

$$200x-275y=0,\quad 200x-136y=1112.$$

By subtraction $-139y=-1112$, or $y=8$, &c.

9. Multiply by 19 and 3 respectively; then

$$57x-209y=0,\quad 57x-57y=24.$$

By subtraction $152y=24$; $\therefore y=\frac{3}{19}$, &c.

10. Multiply the second equation by 2. Then $\frac{x}{2}+\frac{y}{3}=1$ and $\frac{x}{2}-\frac{4y}{3}=6$.

Hence by subtraction $\frac{5}{3}y=-5$; $\therefore y=-3$, &c.

11. Multiply by 3 and 5 respectively; then $x-\frac{y}{2}=\frac{3}{2}$, $x-\frac{3y}{2}=\frac{5}{2}$. By subtraction, $y=-1$, &c.

12. Multiply by 3 and 5 respectively; then $x+9y+42=0$, $x+25y+20=0$. By subtraction, $16y-22=0$; $\therefore\ y=\frac{11}{8}$, &c.

13. Multiply both equations by 5; then $x+25y=-20$, $y+25x=20$. By addition, $26x+26y=0$, $\therefore\ x+y=0$. Hence $24y=-20$; $\therefore\ y=-\frac{5}{6}$, &c.

14. Clear of fractions; then the equations reduce to $x+12y=4$ and $2y-11x=11$. Multiply the first by 11 and add to the second; then

$$134y=55;\ \therefore\ y=\frac{55}{134},\ \text{\&c.}$$

15. The equations reduce to $14x+26y=40$, $11x-20y=-9$. Multiply by 10 and 13 respectively; then $140x+260y=400$, $143x-260y=-117$. By addition $283x=283$; $\therefore\ x=1$, &c.

16. The equations reduce to $3x+y=12$, $4x-9y=47$. Multiply the first equation by 9 and add to the second; then $31x=155$; $\therefore\ x=5$, &c.

17. The equations reduce to $3x-4y=-17$, $4x-3y=-7$. Multiply the first equation by 3 and the second by 4, and subtract; then

$$-7x=-23;\ \therefore\ x=\frac{23}{7},\ \text{\&c.}$$

18. The equations reduce to $4x-3y=14$, $14x+10y=90$.
$\therefore\ 40x-30y=140$, $42x+30y=270$. Hence $82x=410$; $\therefore\ x=5$, &c.

19. The equations reduce to $2x-3y=19$, $32x+3y=151$. Hence

$$34x=170;\ \therefore\ x=5,\ \text{\&c.}$$

XIX. Page 79.

1. By addition $40x+40y=160$; $\therefore\ x+y=4$.
By subtraction $-6x+6y=12$; $\therefore\ -x+y=2$.
Whence $x=1$, $y=3$.

2. By addition $34x+34y=68$; $\therefore\ x+y=2$.
By subtraction $-4x+4y=-32$; $\therefore\ -x+y=-8$.
Whence $x=5$, $y=-3$.

3. By addition $65x-65y=130$; $\therefore\ x-y=2$.
By subtraction $37x+37y=444$; $\therefore\ x+y=12$.
Whence $x=7$, $y=5$.

4. By addition $42x+42y=126$; $\therefore\ x+y=3$.
By subtraction $16x+128y=-64$; $\therefore\ x+8y=-4$.
Whence $7y=-7$; $\therefore\ y=-1$, &c.

5. Clear of decimals; then $10x+2y=3$, $1700x+10y=345$. Multiply the first equation by 5; then by subtraction $1650x=330$; $\therefore\ x=\frac{1}{5}$, &c.

6. Clear of decimals; then $50x+100y=275$, or $2x+4y=11$,
and $340x+2y=175$, or $680x+4y=350$.
By subtraction $678x=339$; $\therefore\ x=\frac{1}{2}$, &c.

7. Since $3x-4y+2=5x-6y-2$, $-2x+2y=-4$.
Since $3x-4y+2=7x+2y+4$, $-4x-6y=2$.
Whence $x=1$, $y=-1$.

8. Since $4x-6y-3=7x+2y-4$, $-3x-8y=-1$.
Since $4x-6y-3=-2x+3y+24$, $6x-9y=27$.
Whence $x=3$, $y=-1$.

9. Since $3x+\frac{7}{2}y-2=20$, $6x+7y=44$.
Since $11y-\frac{2}{5}x=20$, $55y-2x=100$.
Multiply the last equation by 3; then by addition $172y=344$; $\therefore\ y=2$, &c.

10. Since $5x+\frac{y}{5}-1=4$, $25x+y=25$.
Since $3y+\frac{x}{3}-2=4$, $9y+x=18$.
Multiply the first equation by 9 and subtract the second; then

$$224x=207;\ \therefore\ x=\frac{207}{224},\ \text{\&c.}$$

11. Since $\frac{7+x}{3}=\frac{9+y}{5}$, $5(7+x)=3(9+y)$; $\therefore\ 5x-3y=-8$.
Since $\frac{9+y}{5}=\frac{11+x+y}{7}$, $7(9+y)=5(11+x+y)$; $\therefore\ -5x+2y=-8$.
Whence $y=16$, $x=8$.

12. Since $\frac{x+1}{2}=\frac{y+2}{3}$, $3(x+1)=2(y+2)$; $\therefore\ 3x-2y=1$.
Since $\frac{y+2}{3}=\frac{x+y}{4}$, $4(y+2)=3(x+y)$; $\therefore\ -3x+y=-8$.
Whence $y=7$, $x=5$.

13. Clear of brackets; then, after reduction, the equations become respectively $x-y=0$ and $-x-y=4$. Whence $x=y=-2$.

14. Clear of brackets; then, after reduction, we have $x-5y=-5$ and $x-y=1$. Whence $y=\frac{3}{2}$, $x=\frac{5}{2}$.

15. We have $\frac{18}{x}-\frac{8}{y}=4,\ \frac{18}{x}+\frac{8}{y}=10.$

By addition $\frac{36}{x}=14$, or $36=14x$; $\therefore x=\frac{18}{7}$.

By subtraction $\frac{16}{y}=6$, or $16=6y$; $\therefore y=\frac{8}{3}$.

16. We have $\frac{6}{x}+\frac{15}{y}=21,\ \frac{6}{x}-\frac{4}{y}=22.$

Hence $\frac{19}{y}=-1$; $\therefore y=-19$.

Then $\frac{6}{x}-\frac{15}{19}=21$; $\therefore \frac{6}{x}=\frac{414}{19}$; $\therefore x=\frac{6\times 19}{414}=\frac{19}{69}$.

17. We have $\frac{3}{x}+\frac{4}{y}=3$ and $\frac{12}{x}-\frac{4}{y}=2$.

Hence $\frac{15}{x}=5$, or $15=5x$; $\therefore x=3$.

Then $\frac{3}{3}+\frac{4}{y}=3$, or $\frac{4}{y}=2$; $\therefore y=2$.

18. $\frac{12}{x}-\frac{10}{y}=18,\ \frac{35}{x}-\frac{10}{y}=25.$

Hence $\frac{23}{x}=7$, or $23=7x$; $\therefore x=\frac{23}{7}$.

Then $7\div\frac{23}{7}-\frac{2}{y}=5$, whence $y=-\frac{23}{33}$.

19. $3x+\frac{9}{y}=\frac{21}{2},\ 3x-\frac{2}{y}=\frac{26}{3}.$

Hence $\frac{11}{y}=\frac{21}{2}-\frac{26}{3}=\frac{11}{6}$; $\therefore y=6$.

Then $x+\frac{3}{6}=\frac{7}{2}$; $\therefore x=3$.

20. $8x-\frac{12}{y}=12,\ 8x+\frac{15}{y}=-6.$

Hence $-\frac{27}{y}=18$, or $-27=18y$; $\therefore y=-\frac{3}{2}$.

Then $2x+2=3$; $\therefore x=\frac{1}{2}$.

21. $$\frac{3}{x} - 3y = 8, \quad \frac{21}{x} + 3y = 15.$$

Hence $$\frac{24}{x} = 23; \quad \therefore\ x = \frac{24}{23}.$$

Then $$y = 5 - \frac{7}{x} = 5 - \frac{161}{24} = -\frac{41}{24}.$$

22. $$\frac{14}{x} - 18y = 114, \quad \frac{45}{x} + 18y = 63.$$

Hence $$\frac{59}{x} = 177; \quad \therefore\ x = \frac{59}{177} = \frac{1}{3}.$$

Then $$2y = 7 - \frac{5}{x} = 7 - 15 = -8; \quad \therefore\ y = -4.$$

23. Since $$x - \frac{5}{y} + 7 = 3x - \frac{9}{y} - 11, \quad -2x + \frac{4}{y} = -18.$$

Since $$x - \frac{5}{y} + 7 = 7x + 21, \quad -6x - \frac{5}{y} = 14.$$

Hence $$-6x + \frac{12}{y} = -54 \text{ and } -6x - \frac{5}{y} = 14.$$

$$\therefore\ \frac{17}{y} = -68; \quad \therefore\ y = -\frac{17}{68} = -\frac{1}{4}.$$

Then $$-2x = -18 - \frac{4}{y} = -18 + 16 = -2; \quad \therefore\ x = 1.$$

24. By addition $$2x = 2a + 2b.$$

By subtraction $$2y = 2a - 2b.$$

25. By addition $$2ax = (a+b)^2 + a^2 - b^2 = 2a^2 + 2ab.$$

$$\therefore\ x = (2a^2 + 2ab) \div 2a = a + b.$$

By subtraction $$2by = (a+b)^2 - a^2 + b^2 = 2ab + 2b^2.$$

$$\therefore\ y = (2ab + 2b^2) \div 2b = a + b.$$

26. $$a^2x + aby = a(a^2 + b^2), \quad b^2x + aby = 2ab^2.$$

By subtraction $$(a^2 - b^2)x = a(a^2 - b^2); \quad \therefore\ x = a.$$

Then $$ay = 2ab - bx = 2ab - ab; \quad \therefore\ y = b.$$

27. $$ax - ay = a(a - b), \quad ax - by = 2a^2 - 2b^2.$$

By subtraction $$(b - a)y = -a^2 - ab + 2b^2;$$

$$\therefore\ y = \frac{-a^2 - ab + 2b^2}{-a + b} = a + 2b.$$

Then $$x = a - b + y = a - b + a + 2b = 2a + b.$$

28. $$a^2x+aby=a\,(a^2-b^2),\quad b^2x+aby=b\,(a^2-b^2).$$

By subtraction $$(a^2-b^2)\,x=(a-b)(a^2-b^2);$$
$$\therefore\ x=a-b.$$

Then $$by=a^2-b^2-ax=a^2-b^2-a\,(a-b)=ab-b^2.$$
$$\therefore\ y=(ab-b^2)\div b=a-b.$$

29. $$b^2x-a^2y=0,\quad abx+a^2y=a\,(a+b).$$

By addition $$(b^2+ab)\,x=a\,(a+b);$$
$$\therefore\ x=\frac{a\,(a+b)}{b^2+ab}=\frac{a}{b}.$$

Then $$a^2y=b^2x=b^2\times\frac{a}{b}=ab;$$
$$\therefore\ y=\frac{ab}{a^2}=\frac{b}{a}.$$

30. $$ax+ay=a\,(a+b),\quad ax-by=b^2-a^2.$$

By subtraction $$(a+b)\,y=a^2+ab-b^2+a^2=2a^2+ab-b^2.$$
$$\therefore\ y=\frac{2a^2+ab-b^2}{a+b}=2a-b.$$

Then $$x=a+b-y=a+b-2a+b=2b-a.$$

31. $$b^2x-aby=b^4,\quad a^2x-aby=a^4.$$

Hence $$(b^2-a^2)\,x=b^4-a^4;\quad\therefore\ x=(b^4-a^4)\div(b^2-a^2)=a^2+b^2.$$

Then $$-ay=b^3-bx=b^3-b\,(a^2+b^2)=-ba^2;\quad\therefore\ y=ab.$$

32. $$ax-by=a^2+b^2,\quad bx+by=2ab.$$

Hence $$(a+b)\,x=a^2+2ab+b^2.$$
$$\therefore\ x=(a^2+2ab+b^2)\div(a+b)=a+b.$$

Then $$y=2a-x=2a-(a+b)=a-b.$$

33. $$a^2bx+b^3y=c^2b,\quad a^3x+b^3y=c^3.$$

Hence $$(a^2b-a^3)\,x=c^2b-c^3;\quad\therefore\ x=\frac{c^2b-c^3}{a^2b-a^3}=\frac{c^2\,(b-c)}{a^2\,(b-a)}.$$

Similarly $$y=\frac{c^2\,(a-c)}{b^2\,(a-b)}.$$

34. $$a^2x+aby=a,\quad b^2x+aby=b.$$

Hence $$(a^2-b^2)\,x=a-b;\quad\therefore\ x=\frac{a-b}{a^2-b^2}=\frac{1}{a+b}.$$

Then $$by=1-ax=1-\frac{a}{a+b}=\frac{b}{a+b};\quad\therefore\ y=\frac{1}{a+b}.$$

35. $$\frac{a^2}{x}+\frac{ab}{y}=a,\quad \frac{b^2}{x}+\frac{ab}{y}=b.$$

Hence $$\frac{a^2-b^2}{x}=a-b;\ \therefore\ x=\frac{a^2-b^2}{a-b}=a+b.$$

Then $$\frac{a}{y}=1-\frac{b}{x}=1-\frac{b}{a+b}=\frac{a}{a+b};\ \therefore\ \frac{1}{y}=\frac{1}{a+b},\text{ or }y=a+b.$$

36. Multiply the first equation by $a+c$ and the second by $a+b$. Then,

$$(a+b)(a+c)x+(a+c)^2y=(a+b)(a+c),$$
$$(a+c)(a+b)x+(a+b)^2y=(a+c)(a+b).$$

By subtraction $$\{(a+c)^2-(a+b)^2\}y=0;\ \therefore\ y=0.$$

Then $$(a+b)x=a+b;\ \therefore\ x=1.$$

37. By addition $2bx+2by=4ab;\ \therefore\ x+y=2a.$

By subtraction $2ax-2ay=2ab;\ \therefore\ x-y=b.$

Hence $2x=2a+b$ and $2y=2a-b.$

XX. Page 82.

1. From the sum of the second and third take the first; then

$$z+x+x+y-y-z=18+24-14,$$

that is $$2x=28;\ \therefore\ x=14.$$

Then $$z=18-x=18-14=4,\ y=24-x=24-14=10.$$

2. $$(z+x)+(x+y)-(y+z)=2b+2c-2a,$$

that is $$2x=2b+2c-2a;\ \therefore\ x=b+c-a.$$

Then $$z=2b-x=2b-(b+c-a)=a+b-c.$$
$$y=2c-x=2c-(b+c-a)=c+a-b.$$

3. Multiply the first equation by 2 and subtract from the second; then

$$y-z=2.$$

Multiply the first equation by 4 and subtract from the third; then

$$5y-3z=12.$$

From $$y-z=2\text{ and }5y-3z=12,$$

we find $$y=3,\ z=1.$$

Then $$x=1-y-z=1-3-1=-3.$$

4. We have $10x+6y+14z=4,\ 10x-20y+45z=35.$

Hence $26y-31z=-31$ .... (i).

Also $6x-12y+27z=21,\ 6x+4y+12z=6.$

Hence $-16y+15z=15$ .... (ii).

From (i) and (ii) we get $y=0,\ z=1.$

Then $5x=2-3y-7z=2-7=-5;\ \therefore\ x=-1.$

5. Eliminate y first between I. and II., and then between I. and III.; we then get $$7x+4z=9 \text{ and } 4x-z=-8.$$

Whence $$x=-1,\ z=4.$$

Then from I., $$y=-2.$$

6. Multiply the first equation by 2 and subtract the second; then we have $$z=3.$$

Hence $$x+2y-9=6 \text{ and } 3x-y-15=8,$$

whence $$x=\frac{61}{7} \text{ and } y=\frac{22}{7}.$$

7. Eliminate x from the first and second, and also from the first and third; then in both cases we obtain $y+2z=2$.

The equations are therefore not independent.

This can be seen at once, for on adding the first and third and then dividing by 2 we get the second equation.

Any values of x, y, z which satisfy two of the equations will satisfy the third. If we give to x any value whatever, we can find corresponding values of y and z from two of the equations.

Thus, if $x=0$, $y=2$ and $z=0$; if $x=1$, $y=0$ and $z=1$; if $x=2$, $y=-2$ and $z=2$; &c.

8. Eliminate x between I. and III., and also between II. and III.; we then get $$7y+16z=39 \text{ and } 9y+10z=29,$$
whence $y=1$, $z=2$; and then we find $x=3$.

9. Eliminate x from I. and III., and also from II. and III.; we then get
$$32y-24z=15 \text{ and } 4y-3z=-1, \text{ or } 32y-24z=-8,$$
which is inconsistent with the former equation.

10. First clear of fractions; then we have
$$x+y+z=1,\ 2x+y+16z=4,\ 20x+9y-6z=0.$$
Eliminate y between I. and II., and also between I. and III.; then
$$x+15z=3,\ 11x-15z=-9.$$
Whence $x=-\frac{1}{2}$, $z=\frac{7}{30}$, and then we find $y=\frac{19}{15}$.

11. From I. and II. $ax-cz=0$. Hence from III., $2ax=1$. Thus
$$x=\frac{1}{2a}, \text{ and similarly } y=\frac{1}{2b},\ z=\frac{1}{2c}.$$

12. Multiply in order by a, b, c respectively. Then
$$acy+abz=abc,\ abz+bcx=abc,\ bcx+acy=abc.$$

Subtract the third from the sum of the first and second; then $2abz=abc$. Hence $z=\frac{c}{2}$ and similarly $x=\frac{a}{2}$, $y=\frac{b}{2}$.

XXI. Pages 86—88.

1. Let A have £x and B have £y. Then $x+y=50$; also $\frac{x}{2}+\frac{y}{3}=20$, &c.

2. The two numbers with x, y for digits are $10x+y$ and $10y+x$. Hence $10x+y+9=10y+x$, whence $y-x=1$.

3. Suppose each cow to cost £x and each sheep £y. Then $8x+50y=225$ and $\frac{120}{100}x\times 8+\frac{110}{100}y\times 50=257\frac{1}{2}$, &c.

4. Suppose each cart to carry x tons and each waggon y tons. Then

$$15x+12y=28 \text{ and } 24x+8y=28, \text{ \&c.}$$

5. If A take x days for the whole, he would do $\frac{1}{x}$th each day; and if B take y days for the whole, he would do $\frac{1}{y}$th each day. Hence

$$\frac{1}{x}\times 30+\frac{1}{y}\times 30=1; \text{ also } \frac{1}{x}\times 36+\frac{1}{y}\times 12=1, \text{ \&c.}$$

6. Let the numerator be x and the denominator y.

Then $$\frac{x+1}{y-1}=1, \text{ and } \frac{x+y}{y-x}=4.$$

Hence $$x-y=-2 \text{ and } 5x-3y=0, \text{ \&c.}$$

7. Let the numerator be x and the denominator y.

Then $$\frac{4x}{y+3}=2\frac{x}{y} \text{ and } \frac{x+2}{4y}=\frac{1}{2}\frac{x}{y}.$$

From the first equation we have $2y=y+3$ or $y=3$ [unless $x=0$, which is inadmissible]. Then from the second equation we have $x+2=2x$, whence $x=2$.

8. In the first edition suppose that x and y were the number of pages in the two parts respectively. Then $x+y=600$. Also $x+30=\frac{3}{4}y$, &c.

9. Suppose that A has £x and that B has £y. Then $x+10=2(y-10)$; also $$3(x-10)=y+10, \text{ \&c.}$$

10. Let the prices per bushel of the wheat and barley be x shillings and y shillings respectively.

Then $$30x+50y=375; \text{ also } 50x+30y=385, \text{ \&c.}$$

11. Let the first rectangle be x yards long and y yards wide.

Then $$xy=(x+6)(y-4)=(x+8)(y-5).$$

Hence $$4x-6y=-24 \text{ and } 5x-8y=-40, \text{ \&c.}$$

12. If A take x days for the whole, he would do $\frac{1}{x}$th in a day; and if B take y days for the whole, he would do $\frac{1}{y}$th in a day.

Then $$\frac{15}{x}+\frac{15}{y}=1; \text{ also } \frac{6}{x}+\frac{30}{y}=1, \text{ \&c.}$$

13. Let £x and £y be the stock in the $3\frac{1}{2}$ per cents. and 4 per cents. respectively. Then $$x\times\frac{3\frac{1}{2}}{100}+y\times\frac{4}{100}=120.$$

Also $\frac{x\times 108}{100}+\frac{y\times 120}{100}=3672$. That is $7x+8y=24000$, and
$$108x+120y=367200, \text{ \&c.}$$

14. Let the digits be x and y, y being in the units place. Then
$$10x+y=7\,(x+y);\ \therefore\ 3x=6y.$$

15. Let the digits be x and y, y being the digit in the units place. Then $10x+y=4\,(x+y)$; $\therefore$ $6x=3y$, or $y=2x$. Hence the numbers are 12, 24, 36 and 48.

16. Suppose that A has £x, and that C has £y. Then B has £$\frac{x}{2}$. And D gets £$\left(y-\frac{1}{3}x\right)$. Hence $x+\frac{x}{2}+y+y-\frac{1}{3}x=1000$. Also
$$\frac{x}{2}+100=y+y-\frac{1}{3}x, \text{ \&c.}$$

17. Let x and y be the digits. Then $y=2x$; also
$$10x+y=10y+x-36, \text{ \&c.}$$

18. Let x and y be the digits. Then the numbers are $10x+y$ and $10y+x$. Hence $(10x+y)+(10y+x)=99$, also $(10x+y)-(10y+x)=45$, &c.

19. Let x be the numerator and y be the denominator.

Then $$y=x+12; \text{ also } \frac{x+5}{y+5}=\frac{3}{4}, \text{ \&c.}$$

20. Suppose a man to earn x shillings and a boy y shillings a day. Then $$10x+8y=37; \text{ also } 4x=6y+1, \text{ \&c.}$$

21. Let x and y be the numbers of acres in the two farms. Then $$30\,(x+y)=675\times 20; \text{ also } 25x+20y=500\times 20, \text{ \&c.}$$

22. Let x, y, z be the number of half-crowns, shillings and sixpences respectively. Then $x+y+z=20$; and $\frac{5}{2}x+y+\frac{1}{2}z=20$. Also
$$x+2y+6z=73, \text{ \&c.}$$

23. Suppose the charge to be x francs per mile for a passenger, and y francs per mile for 1 kilo. of extra luggage. Then $200x+5000y=25$, and $150x+1500y=16\frac{1}{2}$; whence $x=\frac{1}{10}$ and $y=\frac{1}{1000}$. Hence a journey of 100 miles with 100 kilos. of luggage (*i.e.* with 75 kilos. extra) will cost in francs

$$100x+7500y=100\times\frac{1}{10}+7500\times\frac{1}{1000}=17\tfrac{1}{2}.$$

24. Let A have x shillings; then he has $2x$ pence. Let B have y shillings; then he has $\frac{y}{2}$ pence.

Then $\qquad 12y+\frac{y}{2}=12x+2x+8$, and $x+y+1=2x+\frac{y}{2}$, &c.

25. Let $4x$ be the number of candidates; then $3x$ pass and x fail. Let y be the number of marks required to pass. Then the average marks of all the $4x$ candidates is $y+2$; hence $4x(y+2)$ marks are obtained altogether. The $3x$ who pass get $y+11$ each, on the average, and therefore $3x(y+11)$ on the whole; also the x who fail get $\frac{y}{2}$ each, on the average, and therefore $x\times\frac{y}{2}$ on the whole. Hence $4x(y+2)=3x(y+11)+\frac{1}{2}xy$, whence $y=50$.

MISCELLANEOUS EXAMPLES. II. Pages 88—91.

A. 1. $\dfrac{2-12-0}{2-3+0}+\dfrac{1+8+27}{1+4-9}=\dfrac{-10}{-1}+\dfrac{36}{-4}=10-9=1.$

2. $2b-3(b-\overline{a-b})-\{2a-3(a-\overline{b-a})\}$
$=2b-3(b-a+b)-\{2a-3(a-b+a)\}=2b-3b+3a-3b-2a+3a-3b+3a$
$=7a-7b.$

3. The first expression $=xy+3x+3y+9-3(xy+x+y+1)$
$\qquad +3(xy-x-y+1)-(xy-3x-3y+9)=0.$

The second expression $=xy+2x+2y+4-4(xy+x+y+1)$
$\qquad +6xy-4(xy-x-y+1)+(xy-2x-2y+4)=0.$

4.

$$\begin{array}{rrrrrrrl}
x^3-3x^2+3x-1\ ) & x^5 & -5x^4 & +7x^3 & -x^2 & -4x & +2 & (\ x^2-2x-2\\
 & x^5 & -3x^4 & +3x^3 & -x^2 & & & \\ \cline{2-7}
 & & -2x^4 & +4x^3 & & -4x & +2 & \\
 & & -2x^4 & +6x^3 & -6x^2 & +2x & & \\ \cline{3-7}
 & & & -2x^3 & +6x^2 & -6x & +2 & \\
 & & & -2x^3 & +6x^2 & -6x & +2 &
\end{array}$$

5. (i) Multiply by 60; then $30x+20(x+1)+15(x+2)+12(x+3)=240$, whence $x=2$.

(ii) Multiply in order by 15 and 9; then

$$5x+3y=75,\ 18x+3y=153;\ \text{whence } x=6, \text{ and } y=15.$$

(iii) Multiply the equations by a, b respectively; then, by subtracting,

$$(a^2-b^2)x=a^3-ab^2+a^2b-b^3;\ \therefore\ x=a+b, \text{ and then } y=a+b.$$

6. Let x be the number of marbles. Then the first boy had $\frac{x}{2}-10$; the second had $\frac{1}{4}\left(\frac{x}{2}+10\right)+10$; and the third had 20.

$$\text{Hence } x=\left\{\frac{x}{2}-10\right\}+\left\{\frac{1}{4}\left(\frac{x}{2}+10\right)+10\right\}+20, \text{ whence } x=60.$$

B. 1. $(-1)^2+\left(\frac{3}{2}\right)^2+\left(-\frac{1}{2}\right)^2+\left(\frac{3}{2}\right)\left(-\frac{1}{2}\right)(-1)=1+\frac{9}{4}+\frac{1}{4}+\frac{3}{4}=4\tfrac{1}{4}$.

2. $(2a-b)^2+(a-2b)^2-4(a-b)^2$

$$=4a^2-4ab+b^2+a^2-4ab+4b^2-4a^2+8ab-4b^2=a^2+b^2.$$

3. $n(n-1)(n-2)+3n(n-1)+n=(n^3-3n^2+2n)+3(n^2-n)+n=n^3$.

$n(n-1)(n-2)(n-3)+6n(n-1)(n-2)+7n(n-1)+n$

$$=n^4-6n^3+11n^2-6n+(6n^3-18n^2+12n)+(7n^2-7n)+n=n^4.$$

4. The expression must $=(x^2-2x+1)(x^2+2x+1)+x+1$

$$=(x^2+1)^2-4x^2+x+1=x^4-2x^2+x+2.$$

5. (i) $3(x^2+6x+9)+5(x^2+10x+25)=8(x^2+16x+64)$,

whence $60x=-360$, or $x=-6$.

(ii) The equations are $\frac{x}{6}+\frac{5y}{6}=8$ and $\frac{x}{12}+\frac{7y}{12}=11$;

$$\therefore\ x+5y=48 \text{ and } x+7y=132, \text{ \&c.}$$

(iii) Multiply the first equation by 2 and add the second; then $\frac{9}{x}=2\tfrac{1}{6}$, $\therefore\ \frac{1}{x}=\frac{13}{54}$, or $x=\frac{54}{13}$. Then $\frac{4}{y}=1-\frac{1}{x}=1-\frac{13}{54}$, whence $y=\frac{216}{41}$.

6. Let x be the number of sovereigns, then $130-x$ will be the number of crowns. Hence $x+\frac{1}{4}(130-x)=$ value of the coins in pounds $=100$, &c.

C. 1. $a-[a+b-\{a+b+c-a-b-c-d\}]$

$$=a-[a+b-\{-d\}]=a-[a+b+d]=a-a-b-d=-b-d.$$

2. The sum $=a+b+\left(\frac{a}{2}-\frac{b}{2}\right)=\frac{3}{2}a+\frac{b}{2}$.

The difference $=a+b-\left(\frac{a}{2}-\frac{b}{2}\right)=\frac{a}{2}+\frac{3}{2}b$.

The product $=(a+b)\times\frac{1}{2}(a-b)=\frac{1}{2}(a+b)(a-b)=\frac{1}{2}(a^2-b^2)$.

3. The expression on the left of (i)

$$=ab-3(ab-a-b+1)+3(ab-2a-2b+4)-(ab-3a-3b+9)=0.$$

The expression on the left of (ii) is

$$ab-4(ab-a-b+1)+6(ab-2a-2b+4)$$
$$-4(ab-3a-3b+9)+(ab-4a-4b+16)=0.$$

4. By Art. 70, Ex. 2,

$$(a^3+b^3+c^3-3abc)\div(a+b+c)=a^2+b^2+c^2-bc-ca-ab.$$

Putting $2x$ for a, $2y$ for b, and z for c, we have

$$(8x^3+8y^3+z^3-12xyz)\div(2x+2y+z)=4x^2+4y^2+z^2-2yz-2zx-4xy.$$

5. (i) The equations are $3x-5y=2$, $x-2y=1$. Multiply the second equation by 3 and subtract from the first; then $y=-1$. Then

$$x=1+2y=1-2=-1.$$

(ii) Multiply by a, b respectively; then, by subtraction,

$$(a^2-b^2)\,x=a^4-b^4;\ \therefore\ x=a^2+b^2.$$

Then $\quad by=a^3-a(a^2+b^2)=-ab^2;\ \therefore\ y=-ab.$

6. Let A make x hits and $30-x$ misses. Then B will make $2x$ hits and $30-2x$ misses. Hence $30-x=3(30-2x)$, &c.

D. 1. $-15\{3+(-15)+0+0\}\div(4)(-4)(1)=-15(-12)\div(-16)$

$$=\frac{15\times 12}{-16}=-\frac{45}{4}.$$

2. $a^2+2ab+b^2-[2a^2-\{a^2+ab-2b^2-ab+b^2\}]$

$$=a^2+2ab+b^2-[2a^2-a^2-ab+2b^2+ab-b^2]$$
$$=a^2+2ab+b^2-2a^2+a^2+ab-2b^2-ab+b^2=2ab.$$

3. By Art. 57, Ex. 1,

$$(x+a)(x+b)(x+c)=x^3+(a+b+c)\,x^2+(bc+ca+ab)\,x+abc.$$

Change the signs of a, b and c; then altering x into a, a into $-x$, b into $-y$, and c into $-z$, we have

$$(a-x)(a-y)(a-z)=a^3-a^2(x+y+z)+a(xy+xz+yz)-xyz.$$

4. Dividing each term by $5a^2bc^2$ we have $15b^2c^3-3ab^3c$.

$$\begin{array}{l}
a-b+2c\ )\ a^2+a(\,b-\ c)-2b^2+7bc-6c^2\ (\ a+2b-3c \\
\qquad\qquad\ \ \underline{a^2-a(\,b-2c)} \\
\qquad\qquad\qquad\quad a(2b-3c) \\
\qquad\qquad\qquad\quad a(2b-3c)-(b-2c)(2b-3c) \\
-(b-2c)(2b-3c)=-(2b^2-7bc+6c^2)=-2b^2+7bc-6c^2.
\end{array}$$

5. (i) Multiply by 12; then

$$6(x-1)-4(x-2)+3(x-3)=0;\ \therefore\ 5x-7=0.$$

(ii) From $\frac{x-3}{2}=\frac{y-1}{3}$ we have $3x-2y=7$.

From $\frac{y-1}{3}=\frac{x+y}{9}$ we have $6y-3x=9$.

Hence, by addition, $4y=16$, &c.

(iii) Multiply the equations by 4 and 5 respectively; then by addition $\frac{47}{x}=188$; $\therefore\ x=\frac{47}{188}=\frac{1}{4}$. Then $\frac{5}{y}=22-\frac{3}{x}=10$; $\therefore\ y=\frac{5}{10}$.

6. Let each side be x feet. Then $x^2-(x-1)^2=69$, that is $2x-1=69$. Whence $x=35$.

E. 1. $b^2+\{ab-bc+ac\}-a\{b-a+c\}+b(c-b+a)-ab$

$$=b^2+ab-bc+ac-ab+a^2-ac+bc-b^2+ab-ab=a^2.$$

Or thus. The first expression $=\{(x+y)+(y+z)+(z+x)\}^2=\{2(x+y+z)\}^2$.

2. The required expression must be the sum of all four expressions. Thus

$$\begin{array}{cccccc}
3a^2+ & 2ab- & ac & & & \\
 & -ab & & +3b^2+ & 2bc & \\
 & & +2ac & & -bc+ & 3c^2 \\
a^2 & & & +b^2 & & +c^2 \\
\hline
4a^2+ & ab+ & ac+ & 4b^2+ & bc+ & 4c^2
\end{array}$$

3. $x^2+2xy+y^2+y^2+2yz+z^2+z^2+2zx+x^2+2(x^2+xy+xz+yz)$
$+2(y^2+xy+xz+yz)+2(z^2+xy+xz+yz)$
$=4x^2+4y^2+4z^2+8xy+8xz+8yz=4(x+y+z)^2$.

4.

$$\begin{array}{l}
\frac{1}{3}x+2\,\Big)\,\frac{1}{3}x^3+\frac{11}{6}x^2-\frac{7}{9}x+\frac{4}{3}\,\Big(\,x^2-\frac{1}{2}x+\frac{2}{3} \\
\qquad\qquad \underline{\frac{1}{3}x^3+\ 2x^2\qquad\qquad} \\
\qquad\qquad\qquad -\frac{1}{6}x^2-\frac{7}{9}x \\
\qquad\qquad\qquad \underline{-\frac{1}{6}x^2-\ x\qquad} \\
\qquad\qquad\qquad\qquad\quad \frac{2}{9}x+\frac{4}{3} \\
\qquad\qquad\qquad\qquad\quad \frac{2}{9}x+\frac{4}{3}
\end{array}$$

$$\begin{array}{l}
ax-b\ )\ acx^3+(ad-bc)x^2-(ac+bd)x+bc\ (\ cx^2+dx-c \\
\qquad\quad \underline{acx^3-bcx^2\qquad\qquad\qquad} \\
\qquad\qquad\qquad adx^2-(ac\ +bd)x \\
\qquad\qquad\qquad \underline{adx^2-bdx\qquad\quad} \\
\qquad\qquad\qquad\qquad\quad -acx+bc \\
\qquad\qquad\qquad\qquad\quad -acx+bc
\end{array}$$

5. (i) $x^2+x-2=x^2+3$; $\therefore x=5$.

(ii) Multiply the second by 2 and subtract the first; then

$$8x=3{\cdot}6-1{\cdot}2=2{\cdot}4;\ \therefore x={\cdot}3.$$

Then $${\cdot}4y=1{\cdot}2-{\cdot}6={\cdot}6,\ \therefore y=1{\cdot}5.$$

(iii) $(a-b)\,x-(a+b)\,y=0$ and $(a-b)\,x+(a-b)\,y=2a\,(a-b)$.

Hence, by subtraction, $2ay=2a\,(a-b)$; $\therefore y=a-b$. Then

$$x=2a-y=2a-(a-b)=a+b.$$

6. Let x and y be the numbers of which x is the larger. Then

$$2\,(x-y)=y+1=x-2.$$

Hence $$2x-3y=1 \text{ and } y-x=-3, \text{ \&c.}$$

F. 1. $3\,(3-5+7)^3-\{3^3+(-5)^3+7^3\}=3\times125-(27-125+343)=130$.

2. The factors can be taken in any order.

Hence product $=(a-x)\,(a+x)\,(a^2+x^2)\,(a^4+x^4)=(a^2-x^2)\,(a^2+x^2)\,(a^4+x^4)$

$$=(a^4-x^4)\,(a^4+x^4)=a^8-x^8.$$

3. (i) $1-a-b+ab+a-ab+b=1$.

(ii) By actual multiplication the expression on the left is

$$1-a-b-c+ab+ac+bc-abc+(a-ab-ac+abc)+(b-bc)+c=1.$$

Or thus. The first two terms $=(1-b)\,(1-c)\,\{(1-a)+a\}=(1-b)\,(1-c)$.

Add the third term; then the sum $=(1-c)\,\{(1-b)+b\}=1-c$.

Hence the whole sum is $1-c+c=1$.

4.

$$\begin{array}{rl}
2y^2+3y-1\ )\ & 4y^4-9y^2+6y-1\ (\ 2y^2-3y+1 \\
& \underline{4y^4+6y^3-2y^2} \\
& \quad -6y^3-7y^2+6y \\
& \quad \underline{-6y^3-9y^2+3y} \\
& \qquad\qquad 2y^2+3y-1 \\
& \qquad\qquad 2y^2+3y-1
\end{array}$$

$$\begin{array}{rl}
a+b-c\ )\ & a^2+3ab+2ac+2b^2+bc-3c^2\ (\ a+2b+3c \\
& \underline{a^2+ab-ac} \\
& \quad 2ab+3ac \\
& \quad \underline{2ab\qquad +2b^2-2bc} \\
& \qquad\quad 3ac\qquad +3bc-3c^2 \\
& \qquad\quad 3ac\qquad +3bc-3c^2
\end{array}$$

5. (i) $ab+ax+bx+x^2=cd+cx+dx+x^2$;

$$\therefore (a+b-c-d)\,x=cd-ab, \text{ \&c.}$$

(ii) Multiply by 7 and 3 respectively, and subtract; then

$$-22y=22;\ \therefore y=-1.$$

Then $$3x=4+7y=4-7=-3;\ \therefore x=-1.$$

6. Let the prices of tea and coffee per pound be x shillings and y shillings respectively.

Then $$12x+3y=35 \text{ and } 3x+12y=27\tfrac{1}{2}, \text{ \&c.}$$

XXIII. PAGE 94.

1. $(2x)^2 + 2(2x) + 1^2 = (2x+1)^2$. 2. $(3x)^2 - 2(3x) + 1^2 = (3x-1)^2$.

3. $1^2 - 2(4x^2) + (4x^2)^2 = (1-4x^2)^2$.

4. $(2a)^2 - 2(2a)(3b) + (3b)^2 = (2a-3b)^2$.

5. $(3a^2)^2 + 2(3a^2)(4b^2) + (4b^2)^2 = (3a^2+4b^2)^2$.

6. $x^2 + 2x\left(\frac{1}{2}y\right) + \left(\frac{1}{2}y\right)^2 = \left(x + \frac{1}{2}y\right)^2$.

7. $(2ax)^2 + 2(2ax)(by) + (by)^2 = (2ax+by)^2$.

8. $(5a^2x)^2 - 2(5a^2x)(3b^2y) + (3b^2y)^2 = (5a^2x - 3b^2y)^2$.

9. $3(a^2+2ab+b^2) = 3(a+b)^2$. 10. $5(a^4 - 2a^2b + b^2) = 5(a^2-b)^2$.

11. $a(a^2 - 6ab + 9b^2) = a(a-3b)^2$.

12. $3a^3(a^2 - 10ab^3 + 25b^6) = 3a^3\{a^2 - 2a(5b^2) + (5b^2)^2\} = 3a^3(a-5b^2)^2$.

13. $-(x^4 - 4x^2y^2 + 4y^4) = -\{(x^2)^2 - 2x^2(2y^2) + (2y^2)^2\} - \{x^2 - 2y^2\}^2$.

14. $-4(x^4 - 2x^2 + 1) = -4\{(x^2)^2 - 2x^2 + 1^2\} = -4(x^2-1)^2$.

15. $xy(4y^2 - 4xy + x^2) = xy(2y-x)^2$.

16. $xy\left(xy + x^2 + \frac{1}{4}y^2\right) = xy\left\{x^2 + 2x\left(\frac{1}{2}y\right) + \left(\frac{1}{2}y\right)^2\right\} = xy\left(x + \frac{y}{2}\right)^2$.

17. $(a+b)^2 + 2(a+b)(2c) + (2c)^2 = \{(a+b)+2c\}^2$.

18. $\{(x^2+y^2) - z^2\}^2$.

19. $(2xy)^2 + 2(2xy)(a+b) + (a+b)^2 = \{2xy + (a+b)\}^2$.

20. $\{3(a+b)\}^2 - 2\{3(a+b)\}c + c^2 = \{3(a+b) - c\}^2$.

XXIV. PAGE 95.

1. $a^2 - 3^2 = (a-3)(a+3)$. 2. $4^2 - b^2 = (2-b)(2+b)$.

3. $(5a)^2 - b^2 = (5a-b)(5a+b)$. 4. $x^2 - (3y)^2 = (x-3y)(x+3y)$.

5. $(4x)^2 - (3y)^2 = (4x-3y)(4x+3y)$.

6. $(8a)^2 - (7b)^2 = (8a-7b)(8a+7b)$.

7. $(2a)^2 - (9b)^2 = (2a-9b)(2a+9b)$. 8. $x^2 - (3y^2)^2 = (x-3y^2)(x+3y^2)$.

9. $(6a^2)^2 - (7b)^2 = (6a^2-7b)(6a^2+7b)$.

10. $(2ab)^2 - (3c)^2 = (2ab-3c)(2ab+3c)$.

11. $(3ax)^2 - (7by)^2 = (3ax-7by)(3ax+7by)$.

12. $(7abc)^2 - (6xyz)^2 = (7abc-6xyz)(7abc+6xyz)$.

13. $x\{(2y)^2 - (3x)^2\} = x(2y-3x)(2y+3x)$.

14. $2a\{(2b)^2 - (3a)^2\} = 2a(2b-3a)(2b+3a)$.

15. $3a^3\{a^2-36\}=3a^3(a^2-6^2)=$&c.

16. $7a^5(1-4a^4)=7a^5\{1^2-(2a^2)^2\}=$&c.

17. $8xy(x^2-4y^2)=8xy\{x^2-(2y)^2\}=$&c.

18. $7ab(c^2-a^2b^2)=7ab(c-ab)(c+ab)$.

19. $(x^4-y^4)=(x^2-y^2)(x^2+y^2)=(x-y)(x+y)(x^2+y^2)$.

20. $(x^2)^2-(4y^2)^2=(x^2-4y^2)(x^2+4y^2)=(x-2y)(x+2y)(x^2+4y^2)$.

21. $(9a^2)^2-(4b^2)^2=(9a^2-4b^2)(9a^2+4b^2)=(3a-2b)(3a+2b)(9a^2+4b^2)$.

22. $$\begin{aligned}(25a^2)^2-(16x^2)^2&=(25a^2-16x^2)(25a^2+16x^2)\\&=(5a-4x)(5a+4x)(25a^2+16x^2).\end{aligned}$$

23. $$\begin{aligned}(x^2y^2)^2-(a^2b^2)^2&=(x^2y^2-a^2b^2)(x^2y^2+a^2b^2)\\&=(xy-ab)(xy+ab)(x^2y^2+a^2b^2).\end{aligned}$$

24. $$\begin{aligned}(a^2b^2)^2-(9c^2d^2)^2&=(a^2b^2-9c^2d^2)(a^2b^2+9c^2d^2)\\&=(ab-3cd)(ab+3cd)(a^2b^2+9c^2d^2).\end{aligned}$$

25. $(9x^2y^2)^2-1^2=(9x^2y^2-1)(9x^2y^2+1)=(3xy-1)(3xy+1)(9x^2y^2+1)$.

26. $$\begin{aligned}(4a^2b^2c^2)^2-1^2&=(4a^2b^2c^2-1)(4a^2b^2c^2+1)\\&=(2abc-1)(2abc+1)(4a^2b^2c^2+1).\end{aligned}$$

27. $4^2-(9x^2y^2)^2=(4-9x^2y^2)(4+9x^2y^2)=(2-3xy)(2+3xy)(4+9x^2y^2)$.

28. $$\begin{aligned}(x^4-y^4)(x^4+y^4)&=(x^2-y^2)(x^2+y^2)(x^4+y^4)\\&=(x-y)(x+y)(x^2+y^2)(x^4+y^4).\end{aligned}$$

29. $$\begin{aligned}(a^4-b^4c^4)(a^4+b^4c^4)&=(a^2-b^2c^2)(a^2+b^2c^2)(a^4+b^4c^4)\\&=(a-bc)(a+bc)(a^2+b^2c^2)(a^4+b^4c^4).\end{aligned}$$

30. $$\begin{aligned}a^2(a^8-1)=a^2(a^4-1)(a^4+1)&=a^2(a^2-1)(a^2+1)(a^4+1)\\&=a^2(a-1)(a+1)(a^2+1)(a^4+1).\end{aligned}$$

31. $\{(a+b)-c\}\{(a+b)+c\}$. **32.** $\{(a+b)-2c\}\{(a+b)+2c\}$.

33. $\{2(x+y)\}^2-1^2=\{2(x+y)-1\}\{2(x+y)+1\}$.

34. $\{3(x-y)\}^2-2^2=\{3(x-y)-2\}\{3(x-y)+2\}$.

35. $\{(x+y)-(x-y)\}\{(x+y)+(x-y)\}=2y\,.\,2x$.

36. $\{(2a+b)-(2b+a)\}\{(2a+b)+(2b+a)\}=(a-b)\times 3(a+b)$.

37. $\{x-(x-y)\}\{x+(x-y)\}=y(2x-y)$.

38. $\{a-(2b-a)\}\{a+(2b-a)\}=(2a-2b)\times 2b=4b(a-b)$.

39. $\{2(a+b)-(a-b)\}\{2(a+b)+(a-b)\}=(a+3b)(3a+b)$.

40. $\{3(x+y)-2(x-y)\}\{3(x+y)+2(x-y)\}=(x+5y)(5x+y)$.

41. $(a^2+b^2-2ab)(a^2+b^2+2ab)=(a-b)^2(a+b)^2$.

42. $\{(a+b+c)-(a-b-c)\}\{(a+b+c)+(a-b-c)\}=2(b+c)\times 2a$.

43. $$\begin{aligned}&\{(3a+b-2c)-(a+3b-c)\}\{(3a+b-2c)+(a+3b-c)\}\\&\qquad=(2a-2b-c)(4a+4b-3c).\end{aligned}$$

44. $\{(a-2b+3c)-(a-c)\}\{(a-2b+3c)+(a-c)\}$
$=(-2b+4c)(2a-2b+2c)=4(-b+2c)(a-b+c).$

45. $\{(3x^2+x-2)-(x^2-x-2)\}\{(3x^2+x-2)+(x^2-x-2)\}$
$=(2x^2+2x)(4x^2-4)=8x(x+1)(x^2-1)=8x(x+1)(x+1)(x-1).$

XXV. Page 96.

1. $a^3-(2b)^3=(a-2b)\{a^2+a(2b)+(2b)^2\}=\&c.$

2. $(2a)^3+b^3=(2a+b)\{(2a)^2-(2a)b+b^2\}=\&c.$

3. $(2a)^3-(5x)^3=(2a-5x)\{(2a)^2+(2a)(5x)+(5x)^2\}=\&c.$

4. $a^3-(5x^2)^3=(a-5x^2)\{a^2+a(5x^2)+(5x^2)^2\}=\&c.$

5. $4(a^3+8b^3)=4\{a^3+(2b)^3\}=4(a+2b)\{a^2-a(2b)+(2b)^2\}=\&c.$

6. $(3x)^3-\left(\frac{1}{2}y\right)^3=\left(3x-\frac{1}{2}y\right)\left\{(3x)^2+(3x)\left(\frac{1}{2}y\right)+\left(\frac{1}{2}y\right)^2\right\}=\&c.$

7. $(xy)^3+\left(\frac{1}{3}ab\right)^3=\left(xy+\frac{1}{3}ab\right)\left\{(xy)^2-(xy)\left(\frac{1}{3}ab\right)+\left(\frac{1}{3}ab\right)^2\right\}=\&c.$

8. $(2a^2b^2)^3+(x^2)^3=(2a^2b^2+x^2)\{(2a^2b^2)^2-2a^2b^2x^2+(x^2)^2\}=\&c.$

9. $2x\left(y^3-\frac{1}{8}x^3\right)=2x\left\{y^3-\left(\frac{1}{2}x\right)^3\right\}$
$=2x\left(y-\frac{1}{2}x\right)\left\{y^2+y\left(\frac{1}{2}x\right)+\left(\frac{1}{2}x\right)^2\right\}=\&c.$

10. $9ab^2\left(a^3-\frac{1}{27}b^3\right)=9ab^2\left\{a^3-\left(\frac{1}{3}b\right)^3\right\}$
$=9ab^2\left(a-\frac{1}{3}b\right)\left\{a^2+a\left(\frac{1}{3}b\right)+\left(\frac{1}{3}b\right)^2\right\}=\&c.$

11. $3ab(a^3+8b^3)=3ab\{a^3+(2b)^3\}=3ab(a+2b)\{a^2-a(2b)+(2b)^2\}=\&c.$

12. $5bc(8a^3-b^3c^3)=5bc\{(2a)^3-(bc)^3\}$
$=5bc(2a-bc)\{(2a)^2+(2a)(bc)+(bc)^2\}=\&c.$

13. $(a^3)^2-8^2=(a^3-8)(a^3+8)=(a-2)(a^2+2a+4)(a+2)(a^2-2a+4).$

14. $(8a^3)^2-(27b^3)^2=(8a^3-27b^3)(8a^3+27b^3)$
$=\{(2a)^3-(3b)^3\}\{(2a)^3+(3b)^3\}$
$=(2a-3b)(4a^2+6ab+9b^2)(2a+3b)(4a^2-6ab+9b^2).$

15. $(x^6)^2-(a^3b^3)^2=(x^6-a^3b^3)(x^6+a^3b^3)$
$=(x^2-ab)(x^4+abx^2+a^2b^2)(x^2+ab)(x^4-abx^2+a^2b^2).$

16. $\{(x+2y)-y\}\{(x+2y)^2+(x+2y)y+y^2\}$
$=(x+y)\{x^2+4xy+4y^2+xy+2y^2+y^2\}=\&c.$

17. $\{(x+2y)+(y+2x)\}\{(x+2y)^2-(x+2y)(y+2x)+(y+2x)^2\}$
$=(3x+3y)\{(x^2+4xy+4y^2)-(2x^2+5xy+2y^2)+(y^2+4xy+4x^2)\}$
$=3(x+y)(3x^2+3xy+3y^2).$

18. $\{(2y-x)-(2x-y)\}\{(2y-x)^2+(2y-x)(2x-y)+(2x-y)^2\}$
$=(3y-3x)\{(4y^2-4xy+x^2)+(-2y^2+5xy-2x^2)+(4x^2-4xy+y^2)\}$
$=3(y-x)\{3x^2-3xy+3y^2\}$.

19. $\{(x-3y)-(y-3x)\}\{(x-3y)^2+(x-3y)(y-3x)+(y-3x)^2\}$
$=(4x-4y)\{(x^2-6xy+9y^2)+(-3x^2+10xy-3y^2)+(y^2-6xy+9x^2)\}$
$=4(x-y)\{7x^2-2xy+7y^2\}$.

20. $\{(2y-x)+(2x-y)\}\{(2y-x)^2-(2y-x)(2x-y)+(2x-y)^2\}$
$=(x+y)\{(4y^2-4xy+x^2)-(-2x^2+5xy-2y^2)+(4x^2-4xy+y^2)\}$
$=(x+y)\{7x^2-13xy+7y^2\}$.

XXVIII. PAGE 105.

1. $(4x^2)^2-(25y^2)^2=(4x^2-25y^2)(4x^2+25y^2)$
$=\{(2x)^2-(5y)^2\}(4x^2+25y^2)=(2x-5y)(2x+5y)(4x^2+25y^2)$.

2. $(9a^2)^2-(4b^2)^2=(9a^2-4b^2)(9a^2+4b^2)=\{(3a)^2-(2b)^2\}(9a^2+4b^2)$
$=(3a-2b)(3a+2b)(9a^2+4b^2)$.

3. $x(x^4-81)=x(x^2-9)(x^2+9)=x(x-3)(x+3)(x^2+9)$.

4. $1+(3a)^3=(1+3a)\{1-3a+(3a)^2\}=$&c.

5. $\{3^3+(2x)^3\}=(3+2x)\{3^2-3(2x)+(2x)^2\}=$&c.

6. $x(27x^3+8)=x\{(3x)^3+2^3\}=x(3x+2)\{(3x)^2-(3x)2+2^2\}=$&c.

7. $\{(a+b)-(c-d)\}\{(a+b)+(c-d)\}=$&c.

8. $\{(a+b)-2(c-d)\}\{(a+b)+2(c-d)\}=$&c.

9. $\{(a+b)-(a+c)\}\{(a+b)+(a+c)\}=$&c.

10. $\{(x+y)^2-(x-y)^2\}\{(x+y)^2+(x-y)^2\}$
$=\{(x+y)-(x-y)\}\{(x+y)+(x-y)\}\{(x^2+2xy+y^2)+(x^2-2xy+y^2)\}$
$=\{2y\}\{2x\}\{2x^2+2y^2\}=8xy(x^2+y^2)$.

11. $\{(a+b+c)-2c\}\{(a+b+c)+2c\}=$&c.

12. $\{(a+b-3c)-3c\}\{(a+b-3c)+3c\}=$&c.

13. $\{(a^2+ab+b^2)-(a^2-b^2)\}\{(a^2+ab+b^2)+(a^2-b^2)\}$
$=\{ab+2b^2\}\{2a^2+ab\}=\{b(a+2b)\}\{a(2a+b)\}=$&c.

14. $\{(a^2+ab+b^2)-(a^2-ab+b^2)\}\{(a^2+ab+b^2)+(a^2-ab+b^2)\}$
$=\{2ab\}\{2a^2+2b^2\}=4ab(a^2+b^2)$.

15. $\{(x^3+3x)-(3x^2+1)\}\{(x^3+3x)+(3x^2+1)\}$
$=(x^3-3x^2+3x-1)(x^3+3x^2+3x+1)=(x-1)^3(x+1)^3$.

16. $\{(x^2+5xy+y^2)-(x^2-xy+y^2)\}\{(x^2+5xy+y^2)+(x^2-xy+y^2)\}$
$=\{6xy\}\{2x^2+4xy+2y^2\}=12xy(x^2+2xy+y^2)=12xy(x+y)^2$.

17. $\{(a+b+c+d)-(a-b+c-d)\}\{(a+b+c+d)+(a-b+c-d)\}$
$=\{2b+2d\}\{2a+2c\}=4(b+d)(a+c)$.

18. $\{(3a+2b+c)-(a+2b+3c)\}\,\{(3a+2b+c)+(a+2b+3c)\}$
$=\{2a-2c\}\,\{4a+4b+4c\}=8\,(a-c)\,(a+b+c).$

19. $\{(2x+3y)+(3x+2y)\}\,\{(2x+3y)^2-(2x+3y)(3x+2y)+(3x+2y)^2\}$
$=\{5x+5y\}\,\{(4x^2+12xy+9y^2)-(6x^2+13xy+6y^2)+(9x^2+12xy+4y^2)\}$
$=\{5\,(x+y)\}\,\{(7x^2+11xy+7y^2)\}.$

20. $\{(a+3b)-(3a+b)\}\,\{(a+3b)^2+(a+3b)\,(3a+b)+(3a+b)^2\}=$&c.

21—26. These should be done by inspection, if possible; when this fails, the required result can always be obtained by the method of Art. 104. We give one example.

$$x^2-\frac{3}{4}xy-\frac{1}{4}y^2=x^2-\frac{3}{4}xy+\left(\frac{3}{8}y\right)^2-\left(\frac{3}{8}y\right)^2-\frac{1}{4}y^2$$
$$=\left(x-\frac{3}{8}y\right)^2-\frac{25}{64}y^2=\left\{\left(x-\frac{3}{8}y\right)-\frac{5}{8}y\right\}\left\{x-\frac{3}{8}y+\frac{5}{8}y\right\}$$
$$=(x-y)\left(x+\frac{1}{4}y\right).$$

27. $(x^2-1)\,(x^2-4)=(x-1)\,(x+1)\,(x-2)\,(x+2).$

28. $(x^2-9y^2)\,(x^2-4y^2)=(x-3y)\,(x+3y)\,(x-2y)\,(x+2y).$

29 and **30** should be done by inspection [Art. 100].

31. $\{(a+b)-3c\}\,\{(a+b)-2c\}.$ **32.** $\{(x+y)-2z\}\,\{(x+y)-5z\}.$

33. $\{(a+b)-3\,(c+d)\}\,\{(a+b)-5\,(c+d)\}.$

34—37. Here the factors will become obvious after some rearrangement of the terms. Group the terms which have the same coefficient, particularly when there is a literal coefficient which only occurs in *one* power. Whenever the expression is of the second degree in any one of the letters involved, the factors can always be found by the method of Art. 104.

34. $x\,(x+2)-y\,(y+2)=x^2-y^2+2\,(x-y)$ [the factor $x-y$ is now obvious]
$=(x-y)\,\{(x+y)+2\}.$

or, as in Art. 104, $x^2+2x+1-(y^2+2y+1)$, that is
$(x+1)^2-(y+1)^2=\{(x+1)-(y+1)\}\,(x+1+y+1)=$&c.

35. $x^2+4x-y^2-4y=x^2-y^2+4\,(x-y)=(x-y)\,\{(x+y)+4\}.$
or, as in Art. 104, $x^2+4x+4-y^2-4y-4=(x+2)^2-(y+2)^2=$&c.

36. $x^3+x-5\,(x^2+1)=x\,(x^2+1)-5\,(x^2+1)=(x^2+1)\,(x-5).$

37. $x^3+x^2-4\,(x+1)=x^2\,(x+1)-4\,(x+1)=(x+1)\,(x^2-4)$
$=(x+1)\,(x-2)\,(x+2).$

38. $2\,(x^3-x)-3\,(x^2-1)=2x\,(x^2-1)-3\,(x^2-1)=(2x-3)\,(x^2-1)$
$=(2x-3)\,(x-1)\,(x+1).$

39. $5(x^3 - x) - (x^2 - 1) = 5x(x^2 - 1) - (x^2 - 1) = (5x - 1)(x^2 - 1)$
$$= (5x - 1)(x - 1)(x + 1).$$

40. $a(x^3 + 1) + x + 1$ [the factor $x + 1$ is now obvious]
$$= (x + 1)\{a(x^2 - x + 1) + 1\}.$$

41. $a(x^3 + 1) + b(x + 1) = (x + 1)\{a(x^2 - x + 1) + b\}.$

42. Arrange in powers of b; then $x^3 - a^2x + b(x^2 - a^2)$
$$= x(x^2 - a^2) + b(x^2 - a^2) = (x + b)(x^2 - a^2) = \&c.$$

43. $b(x^3 + x) + a(x^2 + 1) = bx(x^2 + 1) + a(x^2 + 1) = (bx + a)(x^2 + 1).$

44. $a(x^2 + xy) + b(y^2 + xy) = ax(x + y) + by(x + y) = (ax + by)(x + y).$

45. $a^2(b^2 + 1) + (b^2 + 1) = (b^2 + 1)(a^2 + 1).$

46. $a^2(b^2 - 1) + b^2 - 1 = (b^2 - 1)(a^2 - 1).$

47. $a(c - d) + b(d - c) = a(c - d) - b(c - d) = (a - b)(c - d).$

48. $a(c^2 - d^2) + b(d^2 - c^2) = (a - b)(c^2 - d^2) = \&c.$

49. $x^2 - y^2 + z(x - y) = (x - y)(x + y) + z(x - y) = (x - y)(x + y + z).$

50. $(a - b)(a + b) - (a - b)^2 = (a - b)\{(a + b) - (a - b)\} = 2b(a - b)$;
or thus: we have $a^2 - b^2 - (a^2 - 2ab + b^2) = 2ab - 2b^2 = 2b(a - b).$

51. Since b only occurs to *one* power, the factors become obvious by arranging thus
$$a^4 - c^4 + b^2(a^2 - c^2) = (a^2 - c^2)\{(a^2 + c^2) + b^2\} = \&c.$$

52. $a^2 - b^2 + c(b - a) = (a - b)(a + b) - c(a - b) = (a - b)(a + b - c).$

53. $a^2 - c^2 - (a - c) = (a - c)\{(a + c) - 1\}$;
or, as in Art. 104, $a^2 - a + \frac{1}{4} - c^2 + c - \frac{1}{4} = \left(a - \frac{1}{2}\right)^2 - \left(c - \frac{1}{2}\right)^2$
$$= \left\{\left(a - \frac{1}{2}\right) - \left(c - \frac{1}{2}\right)\right\}\left\{\left(a - \frac{1}{2}\right) + \left(c - \frac{1}{2}\right)\right\} = \&c.$$

54. There is only one power of b. Hence we write
$1 - a^2x^2 + b(x - ax^2) = 1 - a^2x^2 + bx(1 - ax) = (1 - ax)\{(1 + ax) + bx\}.$

55. $1 - a^2x^2 - b(ax^3 - x^2) = 1 - a^2x^2 - bx^2(ax - 1) = (1 - ax)\{(1 + ax) + bx^2\}.$

56. $a^2c^2 + abc + d(ac + b) = ac(ac + b) + d(ac + b) = (ac + b)(ac + d).$

57. $x(a^2 + ab) + y(b^2 + ab) + ac + bc = ax(a + b) + by(a + b) + c(a + b)$
$$= (a + b)(ax + by + c).$$

58. $a^2 - 2ac + c^2 - b^2 + 2bd - d^2 = (a - c)^2 - (b - d)^2 = \&c.$

59. $(2ab)^2-(a^2+b^2-c^2)^2=(2ab+a^2+b^2-c^2)(2ab-a^2-b^2+c^2)$
$=\{(a+b)^2-c^2\}\{c^2-(a-b)^2\}=(a+b+c)(a+b-c)(c+a-b)(c-a+b).$

60. $\{(a^2-b^2+c^2-d^2)+(2ac-2bd)\}\{(a^2-b^2+c^2-d^2)-(2ac-2bd)\}$
$=\{(a+c)^2-(b+d)^2\}\{(a-c)^2-(b-d)^2\}$
$=(a+c+b+d)(a+c-b-d)(a-c+b-d)(a-c-b+d).$

61. $x^4+2x^2+1-25x^2=(x^2+1)^2-25x^2=(x^2+1+5x)(x^2+1-5x).$

62. $x^4+2x^2y^2+y^4-9x^2y^2=(x^2+y^2)^2-(3xy)^2=$&c.

63. $x^4-2x^2y^2+y^2-9x^2y^2=(x^2-y^2)^2-(3xy)^2=$&c.

64. $x^4-2x^2+1-x^2=(x^2-1)^2-x^2=$&c.

65. $\{(x^2+4x)-5\}\{(x^2+4x)+3\}=(x^2+4x-5)(x^2+4x+3)$
$=(x^2+4x-5)(x+1)(x+3).$

66. $(x^2+7x)^2+18(x^2+7x)+72-280$
$=(x^2+7x)^2+18(x^2+7x)+9^2-81+72-280$
$=\{(x^2+7x)+9\}^2-289=\{(x^2+7x+9)+17\}\{x^2+7x+9-17\}$
$=(x^2+7x+26)(x^2+7x-8)=(x^2+7x+26)(x-1)(x+8).$

XXX. PAGE 108.

5. We have $ab^3(a^2+b^2)$ and $a^2(a^4-b^4)=a^2(a^2-b^2)(a^2+b^2).$
Hence the H. C. F. is $a(a^2+b^2).$

6. We have $b(a^2+3b^2)$ and $a^2(a^4-9b^4)=a^2(a^2-3b^2)(a^2+3b^2).$
Hence the H. C. F. is $a^2+3b^2.$

7. We have $a^2x^2(x+2a)$ and $a^2x^2(x^2-4a^2)=a^2x^2(x-2a)(x+2a).$
Hence the H. C. F. is $a^2x^2(x+2a).$

8. We have $a^4x^2(a^2-4x^2)$ and $a^2x^2(a^4-16x^4)=a^2x^2(a^2-4x^2)(a^2+4x^2).$
Hence the H. C. F. is $a^2x^2(a^2-4x^2).$

9. We have $(x+1)(x+2)$ and $(x+2)(x+4).$
Hence the H. C. F. is $x+2.$

10. We have $x(x+y)(x+2y)$ and $x^2(x+2y)(x+4y).$
Hence the H. C. F. is $x(x+2y).$

11. We have $(x-1)(3x-1)$ and $(x-1)(4x-1).$
Hence the H. C. F. is $x-1.$

12. We have $(a-b)(3a-b)$ and $a^2(a-b)(4a-b).$
Hence the H. C. F. is $a-b.$

XXXI. PAGES 114, 115.

1.

$$\begin{array}{r}
x^2-5x+4\ )\ x^3-5x^2+4\ \ (\ x \\
x^3-5x^2+4x \\
\hline
-4\ )\ -4x+4 \\
\hline
x-1\ )\ x^2-5x+4\ (\ x-4 \\
x^2-\ \ x \\
\hline
-4x+4 \\
-4x+4
\end{array}$$

∴ H. C. F. is $x-1$.

2.

$$x^4-5x^3y+4xy^3=x(x^3-5x^2y+4y^3)$$

$$\begin{array}{r}
x^2-5xy+4y^2\ )\ x^3-5x^2y+4y^3\ (\ x \\
x^3-5x^2y+4xy^2 \\
\hline
-4y^2\ )\ -4xy^2+4y^3 \\
\hline
x\ \ -\ \ y\ )\ x^2-5xy+4y^2\ (\ x-4y \\
x^2-\ \ xy \\
\hline
-4xy+4y^2 \\
-4xy+4y^2
\end{array}$$

∴ H.C.F. is $x-y$.

3.

$$\begin{array}{r}
2x^2-5x+2\ )\ 4x^3+12x^2-\ \ \ x-3\ (\ 2x+11 \\
4x^3-10x^2+\ \ 4x \\
\hline
22x^2-\ \ 5x-\ \ 3 \\
22x^2-55x+22 \\
\hline
25\ )\ 50x-25 \\
\hline
2x-\ \ 1\ )\ 2x^2-5x+2\ (\ x-2 \\
2x^2-\ \ x \\
\hline
-4x+2 \\
-4x+2
\end{array}$$

∴ H. C. F. is $2x-1$.

4.

$$\begin{array}{r}
2x^2-5xy+2y^2\ )\ 4x^3+12x^2y-\ \ \ xy^2-3y^3\ (\ 2x+11y \\
4x^3-10x^2y+\ \ 4xy^2 \\
\hline
22x^2y-\ \ 5xy^2-\ \ 3y^3 \\
22x^2y-55xy^2+22y^3 \\
\hline
25y^2\ )\ 50xy^2-25y^3 \\
\hline
2x\ \ -\ \ y\ )\ 2x^2-5xy+2y^2\ (\ x-2y \\
2x^2-\ \ xy \\
\hline
-4xy+2y^2 \\
-4xy+2y^2
\end{array}$$

∴ H. C. F. is $2x-y$.

5.

$$\begin{array}{r}
x^4-3x^2+2\ )\ x^4+3x^2-10\ (\ 1 \\
x^4-3x^2+\ \ 2 \\
\hline
6\ )\ 6x^2-12 \\
\hline
x^2-\ \ 2\ )\ x^4-3x^2+2\ (\ x^2-1 \\
x^4-2x^2 \\
\hline
-\ \ x^2+2 \\
-\ \ x^2+2
\end{array}$$

H. C. F. is x^2-2.

6. $$x^6+3x^4y-10x^2y^2=x^2(x^4+3x^2y-10y^2).$$

$$\begin{array}{l}
x^4-3x^2y+2y^2\ )\ x^4+3x^2y-10y^2\ (\ 1 \\
\qquad\qquad\qquad\quad \underline{x^4-3x^2y+\ \ 2y^2} \\
\qquad\qquad\qquad\quad 6y\ )\ \underline{6x^2y-12y^2} \\
\qquad\qquad\qquad\qquad\qquad x^2\ -\ 2y\ )\ x^4-3x^2y+2y^2\ (\ x^2-y \\
\qquad\qquad\qquad\qquad\qquad\qquad\qquad\quad \underline{x^4-2x^2y} \\
\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad -\ x^2y+2y^2 \\
\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad -\ x^2y+2y^2
\end{array}$$

H.C.F. is x^2-2y.

7.
$$\begin{array}{l}
2a^2-5a+2\ )\ 2a^3-3a^2-\ 8a+12\ (\ a+1 \\
\qquad\qquad\qquad \underline{2a^3-5a^2+\ 2a} \\
\qquad\qquad\qquad\qquad 2a^2-10a+12 \\
\qquad\qquad\qquad\qquad \underline{2a^2-\ 5a+\ 2} \\
\qquad\qquad\qquad -5\ )\ \underline{-\ 5a+10} \\
\qquad\qquad\qquad\qquad\qquad a-\ 2\ )\ 2a^2-5a+2\ (\ 2a-1 \\
\qquad\qquad\qquad\qquad\qquad\qquad\quad \underline{2a^2-4a} \\
\qquad\qquad\qquad\qquad\qquad\qquad\qquad -\ a+2 \\
\qquad\qquad\qquad\qquad\qquad\qquad\qquad -\ a+2
\end{array}$$

H.C.F. is $a-2$.

8.
$$\begin{array}{l}
2b^2-5b+2\ )\ 12b^3-\ 8b^2-\ 3b+2\ (\ 6b+11 \\
\qquad\qquad\qquad \underline{12b^3-30b^2+12b} \\
\qquad\qquad\qquad\qquad 22b^2-15b+\ 2 \\
\qquad\qquad\qquad\qquad \underline{22b^2-55b+22} \\
\qquad\qquad\qquad\qquad 20\ )\ \underline{40b-20} \\
\qquad\qquad\qquad\qquad\qquad 2b-\ 1\ )\ 2b^2-5b+2\ (\ b-2 \\
\qquad\qquad\qquad\qquad\qquad\qquad\quad \underline{2b^2-\ b} \\
\qquad\qquad\qquad\qquad\qquad\qquad\qquad -4b+2 \\
\qquad\qquad\qquad\qquad\qquad\qquad\qquad -4b+2
\end{array}$$

H.C.F. is $2b-1$.

9.
$$\begin{array}{l}
x^2-3x+2\ )\ x^3-3x\ +2\ \ (\ x+3 \\
\qquad\qquad\qquad \underline{x^3-3x^2+2x} \\
\qquad\qquad\qquad\quad 3x^2-5x+2 \\
\qquad\qquad\qquad\quad \underline{3x^2-9x+6} \\
\qquad\qquad\qquad\qquad 4\ )\ \underline{4x-4} \\
\qquad\qquad\qquad\qquad\qquad x-1\ )\ x^2-3x+2\ (\ x-2 \\
\qquad\qquad\qquad\qquad\qquad\qquad\quad \underline{x^2-\ x} \\
\qquad\qquad\qquad\qquad\qquad\qquad\quad -2x+2 \\
\qquad\qquad\qquad\qquad\qquad\qquad\quad -2x+2
\end{array}$$

Hence $x-1$ is the H.C.F.

10.
$$\begin{array}{l}
x^4y^4-3x^2y^2+2\ )\ x^6y^6-3x^2y^2+2\qquad (\ x^2y^2+3 \\
\qquad\qquad\qquad\qquad \underline{x^6y^6-3x^4y^4+2x^2y^2} \\
\qquad\qquad\qquad\qquad\quad 3x^4y^4-5x^2y^2+2 \\
\qquad\qquad\qquad\qquad\quad \underline{3x^4y^4-9x^2y^2+6} \\
\qquad\qquad\qquad\qquad\qquad 4\ )\ \underline{4x^2y^2-4} \\
\qquad\qquad\qquad\qquad\qquad\quad x^2y^2-1\ )\ x^4y^4-3x^2y^2+2\ (\ x^2y^2-2 \\
\qquad\qquad\qquad\qquad\qquad\qquad\qquad\quad \underline{x^4y^4-\ x^2y^2} \\
\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad -2x^2y^2+2 \\
\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad -2x^2y^2+2
\end{array}$$

Hence x^2y^2-1 is the H.C.F.

11.

$$\begin{array}{l}
x^3-3a^2x-2a^3\;)\;x^3-\;ax^2-4a^3\quad(\,1 \\
\qquad\qquad\qquad\quad x^3-3a^2x-2a^3 \\
\hline
\qquad\qquad\quad -a\;)\;-\;ax^2+3a^2x-2a^3 \\
\hline
\qquad\qquad\qquad\quad x^2-3ax+2a^2\;)\;x^3-3a^2x-2a^3\quad(\,x+3a \\
\qquad\qquad\qquad\qquad\qquad\qquad\qquad x^3-3ax^2+2a^2x \\
\hline
\qquad\qquad\qquad\qquad\qquad\qquad\qquad\quad 3ax^2-5a^2x-2a^3 \\
\qquad\qquad\qquad\qquad\qquad\qquad\qquad\quad 3ax^2-9a^2x+6a^3 \\
\hline
\qquad\qquad\qquad\qquad\qquad\qquad\qquad\quad 4a^2\;)\;4a^2x-8a^3 \\
\hline
\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\quad x-2a\;)\;x^2-3ax+2a^2\;(\,x-a \\
\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad x^2-2ax \\
\hline
\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad -\;ax+2a^2 \\
\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad -\;ax+2a^2
\end{array}$$

Hence $x-2a$ is the H. C. F.

12.

$$\begin{array}{l}
2a^3+3a^2b-b^3\;)\;4a^3+\;ab^2-\;b^3\;(\,2 \\
\qquad\qquad\qquad\quad 4a^3+6a^2b-2b^3 \\
\hline
\qquad\qquad\quad -b\;)\;-6a^2b+ab^2+b^3 \\
\hline
\qquad\qquad\qquad\quad 6a^2-ab-b^2\;)\;2a^3+3a^2b-\;b^3 \\
\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad 3 \\
\hline
\qquad\qquad\qquad\qquad\qquad\qquad 6a^3+9a^2b-3b^3\;(\,a \\
\qquad\qquad\qquad\qquad\qquad\qquad 6a^3-\;a^2b-\;ab^2 \\
\hline
\qquad\qquad\qquad\qquad\qquad\qquad\quad 10a^2b+\;ab^2-3b^3 \\
\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad 3 \\
\hline
\qquad\qquad\qquad\qquad\qquad\qquad\quad 30a^2b+3ab^2-9b^3\;(5b \\
\qquad\qquad\qquad\qquad\qquad\qquad\quad 30a^2b-5ab^2-5b^3 \\
\hline
\qquad\qquad\qquad\qquad\qquad\qquad\quad 4b^2\;)\;8ab^2-4b^3 \\
\hline
\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad 2a-\;b\;)\;6a^2-\;ab-b^2\;(\,3a+b \\
\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\quad 6a^2-3ab \\
\hline
\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad 2ab-b^2 \\
\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad 2ab-b^2
\end{array}$$

Hence $2a-b$ is the H. C. F.

13.

$$\begin{array}{l}
a^3+b^3 \,)\, a^4+a^2b^2+b^4 \,(\, a \\
\qquad a^4+ab^3 \\
\hline
b^2 \,)\, a^2b^2-ab^3+b^4 \\
\hline
a^2-ab+b^2 \,)\, a^3+b^3 \,(\, a+b \\
\qquad a^3-a^2b+ab^2 \\
\hline
\qquad a^2b-ab^2+b^3 \\
\qquad a^2b-ab^2+b^3
\end{array}$$

Hence the H. C. F. is a^2-ab+b^2.

14.

$$\begin{array}{l}
8a^3+1 \,)\, 16a^4+4a^2+1 \,(\, 2a \\
\qquad 16a^4+2a \\
\hline
4a^2-2a+1 \,)\, 8a^3+1 \,(\, 2a+1 \\
\qquad 8a^3-4a^2+2a \\
\hline
\qquad 4a^2-2a+1 \\
\qquad 4a^2-2a+1
\end{array}$$

Hence the H. C. F. is $4a^2-2a+1$.

15.

$$\begin{array}{l}
x^3-2x^2y-xy^2+2y^3 \,)\, x^3+2x^2y-xy^2-2y^3 \,(\, 1 \\
\qquad x^3-2x^2y-xy^2+2y^3 \\
\hline
4y \,)\, 4x^2y-4y^3 \\
\hline
x^2-y^2 \,)\, x^3-2x^2y-xy^2+2y^3 \,(\, x-2y \\
\qquad x^3-xy^2 \\
\hline
\qquad -2x^2y+2y^3 \\
\qquad -2x^2y+2y^3
\end{array}$$

Hence the H. C. F. is x^2-y^2.

16.

$$\begin{array}{l}
-8+x+5x^2+2x^3 \,)\, -8+9x-4x^2+3x^3 \,(\, 1 \\
\qquad -8+x+5x^2+2x^3 \\
\hline
x \,)\, 8x-9x^2+x^3 \\
\hline
8-9x+x^2 \,)\, -8+x+5x^2+2x^3 \,(\, -1-x \\
\qquad -8+9x-x^2 \\
\hline
\qquad -8x+6x^2+2x^3 \\
\qquad -8x+9x^2-x^3 \\
\hline
3x^2 \,)\, -3x^2+3x^3 \\
\hline
-1+x \,)\, 8-9x+x^2 \,(\, -8+x \\
\qquad 8-8x \\
\hline
\qquad -x+x^2 \\
\qquad -x+x^2
\end{array}$$

Hence $x-1$ is the H. C. F.

17.

$$\begin{array}{l}
2x^3-x^2-x-3 \,)\, 3x^3+x^2+x-2 \\
\qquad 2 \\
\hline
\qquad 6x^3+2x^2+2x-4 \,(\, 3 \\
\qquad 6x^3-3x^2-3x-9 \\
\hline
5 \,)\, 5x^2+5x+5 \\
\hline
x^2+x+1 \,)\, 2x^3-x^2-x-3 \,(\, 2x-3 \\
\qquad 2x^3+2x^2+2x \\
\hline
\qquad -3x^2-3x-3 \\
\qquad -3x^2-3x-3
\end{array}$$

Hence x^2+x+1 is the H. C. F.

18.

$$\begin{array}{l}
x^3-4x+15\ )\ x^4+x^2+25\ (\ x \\
\qquad\qquad\quad \underline{x^4-4x^2+15x\qquad} \\
\qquad\qquad\qquad 5\ )\ \underline{5x^2-15x+25} \\
\qquad\qquad\qquad\qquad x^2-3x+5\ )\ x^3-4x+15\ (\ x+3 \\
\qquad\qquad\qquad\qquad\qquad\qquad\qquad \underline{x^3-3x^2+5x\qquad} \\
\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad 3x^2-9x+15 \\
\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad 3x^2-9x+15
\end{array}$$

Hence x^2-3x+5 is the H. C. F.

19.

$$\begin{array}{l}
3x^2-38x+119\ )\ x^3-19x^2+119x-245 \\
\qquad\qquad\qquad\qquad\qquad\qquad\qquad\quad \underline{3} \\
\qquad\qquad\qquad\ \ 3x^3-57x^2+357x-735\ (\ x \\
\qquad\qquad\qquad\ \ \underline{3x^3-38x^2+119x\qquad} \\
\qquad\qquad\qquad\qquad -19x^2+238x-735 \\
\qquad\qquad\qquad\qquad\qquad\qquad\qquad\quad \underline{3} \\
\qquad\qquad\qquad\qquad -57x^2+714x-2205\ (\ -19 \\
\qquad\qquad\qquad\qquad \underline{-57x^2+722x-2261} \\
\qquad\qquad\qquad\quad -8\ )\ \underline{-8x+56} \\
\qquad\qquad\qquad\qquad\qquad\qquad x-7\ )\ 3x^2-38x+119\ (\ 3x-17 \\
\qquad\qquad\qquad\qquad\qquad\qquad\qquad\quad \underline{3x^2-21x\qquad} \\
\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad -17x+119 \\
\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad -17x+119
\end{array}$$

Hence $x-7$ is the H. C. F.

20.

$$\begin{array}{l}
y\ )\ \underline{4x^2y-5xy^2+y^3} \\
\quad 4x^2-5xy+y^2\ )\ 3x^3-3x^2y+xy^2-y^3 \\
\qquad\qquad\qquad\qquad\qquad\qquad\qquad\quad \underline{4} \\
\qquad\qquad\qquad\quad 12x^3-12x^2y+4xy^2-4y^3\ (\ 3x \\
\qquad\qquad\qquad\quad \underline{12x^3-15x^2y+3xy^2\qquad} \\
\qquad\qquad\qquad\qquad\qquad 3x^2y+xy^2-4y^3 \\
\qquad\qquad\qquad\qquad\qquad\qquad\qquad\quad \underline{4} \\
\qquad\qquad\qquad\qquad\quad 12x^2y+4xy^2-16y^3\ (\ 3y \\
\qquad\qquad\qquad\qquad\quad \underline{12x^2y-15xy^2+3y^3} \\
\qquad\qquad\qquad\qquad\qquad 19y^2\ )\ \underline{19xy^2-19y^3} \\
\qquad\qquad\qquad\qquad\qquad\qquad\quad x-y\ )\ 4x^2-5xy+y^2\ (\ 4x-y \\
\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \underline{4x^2-4xy\qquad} \\
\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\quad -xy+y^2 \\
\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\quad -xy+y^2
\end{array}$$

Hence $x-y$ is the H. C. F.

21.

$$\begin{array}{rl}
12x^2-15xy+3y^2\,) & 6x^3-\ 6x^2y+\ 2xy^2-\ 2y^3 \\
 & \qquad\qquad\qquad\qquad\qquad 2 \\
\hline
 & 12x^3-12x^2y+\ 4xy^2-\ 4y^3\,(\,x \\
 & 12x^3-15x^2y+\ 3xy^2 \\
\hline
 & \qquad 3x^2y+\quad xy^2-\ 4y^3 \\
 & \qquad\qquad\qquad\qquad\qquad 4 \\
\hline
 & \qquad 12x^2y+\ 4xy^2-16y^3\,(\,y \\
 & \qquad 12x^2y-15xy^2+\ 3y^3 \\
\hline
 & \qquad 19y^2\,)\ 19xy^2-19y^3 \\
\hline
 & \qquad\qquad\qquad x-y\,)\ 12x^2-15xy+3y^2\,(\,12x-3y \\
 & \qquad\qquad\qquad\qquad\quad 12x^2-12xy \\
\hline
 & \qquad\qquad\qquad\qquad\qquad -\ 3xy+3y^2 \\
 & \qquad\qquad\qquad\qquad\qquad -\ 3xy+3y^2
\end{array}$$

Hence $x-y$ is the H. C. F.

22.

$$\begin{array}{rl}
2\,) & 2x^2-14x+20 \\
\hline
x^2-\ 7x+10\,) & 4x^3-25x^2+20x+25\,(\,4x+3 \\
 & 4x^3-28x^2+40x \\
\hline
 & \qquad 3x^2-20x+25 \\
 & \qquad 3x^2-21x+30 \\
\hline
 & \qquad\qquad x-\ 5\,)\ x^2-7x+10\,(\,x-2 \\
 & \qquad\qquad\qquad\quad x^2-5x \\
\hline
 & \qquad\qquad\qquad\qquad -2x+10 \\
 & \qquad\qquad\qquad\qquad -2x+10
\end{array}$$

Hence $x-5$ is the H. C. F.

23.

$$\begin{array}{l}
1+4x^2+16x^4\ )\ -2+x-16x^3+8x^4\ (\,-2 \\
\qquad\qquad\qquad\quad -2\qquad -8x^2-32x^4 \\
\hline
\qquad\qquad\qquad x\ )\ x+8x^2-16x^3+40x^4 \\
\hline
\qquad\qquad\qquad\quad 1+8x-16x^2+40x^3\ )\ 1+4x^2+16x^4\qquad (\,1-8x \\
\qquad\qquad\qquad\qquad\qquad\qquad\qquad\quad 1+8x-16x^2+40x^3 \\
\hline
\qquad\qquad\qquad\qquad\qquad\qquad\qquad\quad -8x+20x^2-40x^3+16x^4 \\
\qquad\qquad\qquad\qquad\qquad\qquad\qquad\quad -8x-64x^2+128x^3-320x^4 \\
\hline
\qquad\qquad\qquad\qquad\qquad\qquad\qquad\quad 84x^2\ )\ 84x^2-168x^3+336x^4 \\
\hline
\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\quad 1-2x+4x^2\ )\ 1+4x^2+16x^4\ (\,1+2x+4x^2 \\
\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\quad 1-2x+4x^2 \\
\hline
\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad 2x+16x^4 \\
\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad 2x-4x^2+8x^3 \\
\hline
\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\quad 4x^2-8x^3+16x^4 \\
\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\quad 4x^2-8x^3+16x^4
\end{array}$$

Hence the H.C.F. is $1-2x+4x^2$.

24.

$$\begin{array}{l}
a^2-(4x^2-12x+9)\ )\ a^2+2a-(4x^2-8x+3)\ (\,1 \\
\qquad\qquad\qquad\qquad\quad a^2\qquad -(4x^2-12x+9) \\
\hline
\qquad\qquad\qquad\qquad\quad 2\ )\ 2a-(4x-6) \\
\hline
\qquad\qquad\qquad\qquad\qquad a-(2x-3)\ )\ a^2-(4x^2-12x+9)\ (\,a+(2x-3) \\
\qquad\qquad\qquad\qquad\qquad\qquad\qquad\quad a^2-a\,(2x-3) \\
\hline
\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad a\,(2x-3)-(4x^2-12x+9) \\
\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad a\,(2x-3)-(4x^2-12x+9)
\end{array}$$

Hence the H.C.F. is $a-(2x-3)$.

25.

$$\begin{array}{l}
-x^3+3x^2-7x+21\ )\ 2x^4+19x^2+35\qquad (\,-2x-6 \\
\qquad\qquad\qquad\qquad\quad 2x^4-6x^3+14x^2-42x \\
\hline
\qquad\qquad\qquad\qquad\qquad\quad 6x^3+5x^2+42x+35 \\
\qquad\qquad\qquad\qquad\qquad\quad 6x^3-18x^2+42x-126 \\
\hline
\qquad\qquad\qquad\qquad\qquad\quad 23\ )\ 23x^2\qquad +161 \\
\hline
\qquad\qquad\qquad\qquad\qquad\qquad\quad x^2\qquad +7\ )\ -x^3+3x^2-7x+21\ (\,-x+3 \\
\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad -x^3\qquad -7x \\
\hline
\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\quad 3x^2\qquad +21 \\
\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\quad 3x^2\qquad +21
\end{array}$$

Hence the H.C.F. is x^2+7.

26.

$$\begin{array}{rl}
2x^4+9x^3+14x+3\ ) & 3x^4+14x^3+\ 9x+2 \\
 & \qquad\qquad\qquad\quad 2 \\
\hline
 & 6x^4+28x^3+18x+4\ (\ 3 \\
 & 6x^4+27x^3+42x+9 \\
\hline
 & x^3-24x-5\ )\ 2x^4+\ 9x^3+14x+3\ (\ 2x+9 \\
 & \qquad\qquad\qquad\ \ 2x^4-48x^2-10x \\
 & \qquad\qquad\qquad\qquad 9x^3+48x^2+\ 24x+\ 3 \\
 & \qquad\qquad\qquad\qquad 9x^3\qquad\quad -216x-45 \\
 & \qquad\qquad\qquad\qquad 48\ )\ 48x^2+240x+48 \\
 & \qquad\qquad\qquad\qquad\qquad x^2+\ \ 5x+\ 1\ )\ x^3-24x-5\ (\ x-5 \\
 & \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\ x^3+\ 5x^2+x \\
 & \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\ -\ 5x^2-25x-5 \\
 & \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\ -\ 5x^2-25x-5
\end{array}$$

Hence the H.C.F. is x^2+5x+1.

27.

$$\begin{array}{rl}
x^3+3x^2-x-3\ ) & x^4+4x^3-12x\ -9\ (\ x+1 \\
 & x^4+3x^3-\ \ x^2-3x \\
\hline
 & \quad x^3+\ \ x^2-9x-9 \\
 & \quad x^3+\ 3x^2-\ x-3 \\
\hline
 & -2\ )-\ 2x^2-8x-6 \\
 & \qquad x^2+4x+3\ )\ x^3+3x^2-\ x-3\ (\ x-1 \\
 & \qquad\qquad\qquad\qquad x^3+4x^2+3x \\
 & \qquad\qquad\qquad\qquad -\ x^2-4x-3 \\
 & \qquad\qquad\qquad\qquad -\ x^2-4x-3
\end{array}$$

Hence the H.C.F. is x^2+4x+3.

28.
$$\begin{array}{l}
2x^4+3x^3-2x^2+12x+5\ )\ 3x^4+5x^3-7x^2+2x+2 \\
\qquad\qquad\qquad\qquad\qquad\qquad 2 \\
\hline
\qquad 6x^4+10x^3-14x^2+4x+4\ (\ 3 \\
\qquad 6x^4+9x^3-6x^2+36x+15 \\
\hline
\qquad x^3-8x^2-32x-11\ )\ 2x^4+3x^3-2x^2+12x+5\ (\ 2x+19 \\
\qquad\qquad 2x^4-16x^3-64x^2-22x \\
\hline
\qquad\qquad 19x^3+62x^2+34x+5 \\
\qquad\qquad 19x^3-152x^2-608x-209 \\
\hline
\qquad\qquad 214\ )\ 214x^2+642x+214 \\
\hline
\qquad\qquad x^2+3x+1\ )\ x^3-8x^2-32x-11\ (\ x-11 \\
\qquad\qquad\qquad x^3+3x^2+x \\
\hline
\qquad\qquad\qquad -11x^2-33x-11 \\
\qquad\qquad\qquad -11x^2-33x-11
\end{array}$$

Hence the H.C.F. is x^2+3x+1.

29.
$$\begin{array}{l}
y\ )\ y^4-y^3-y^2-2y \\
\hline
\quad y^3-y^2-y-2\ )\ y^3-2y^2+3y-6\ (\ 1 \\
\qquad\qquad y^3-y^2-y-2 \\
\hline
\qquad\qquad -y^2+4y-4\ )\ y^3-y^2-y-2\ (\ -y-3 \\
\qquad\qquad\qquad y^3-4y^2+4y \\
\hline
\qquad\qquad\qquad 3y^2-5y-2 \\
\qquad\qquad\qquad 3y^2-12y+12 \\
\hline
\qquad\qquad\qquad 7\ )\ 7y-14 \\
\hline
\qquad\qquad\qquad y-2\ )\ -y^2+4y-4\ (\ -y+2 \\
\qquad\qquad\qquad\qquad -y^2+2y \\
\hline
\qquad\qquad\qquad\qquad 2y-4 \\
\qquad\qquad\qquad\qquad 2y-4
\end{array}$$

Hence the H.C.F. is $y-2$.

30.

$$
\begin{array}{l}
y^4-3yz^3+20z^4\,)\,5y^4-\ 3y^3z+\ 64z^4\,(\,5 \\
\qquad\qquad\qquad\quad \underline{5y^4-15yz^3+100z^4} \\
\qquad\qquad\qquad\quad -3z\,)\,\underline{-3y^3z+15yz^3-36z^4} \\
\qquad\qquad\qquad\qquad\quad y^3\ -\ 5yz^2+12z^3\,)\,y^4-3yz^3\ +20z^4\ \ (\,y \\
\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \underline{y^4-5y^2z^2+12yz^3} \\
\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad 5z^2\,)\,\underline{5y^2z^2-15yz^3+20z^4} \\
\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\quad y^2\ -\ 3yz\ +\ 4z^2\,)\,y^3-5yz^2+12z^3\ \ (\,y+3z \\
\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\quad \underline{y^3-3y^2z+\ 4yz^2} \\
\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad 3y^2z-9yz^2+12z^3 \\
\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad 3y^2z-9yz^2+12z^3
\end{array}
$$

Hence the H.C.F. is $y^2-3yz+4z^2$.

31.

$$
\begin{array}{l}
2x^3-11x^2+11x+4\,)\,2x^4-\ 3x^3+\ 7x^2-12x-4\,(\,x+4 \\
\qquad\qquad\qquad\qquad\quad \underline{2x^4-11x^3+11x^2+\ 4x} \\
\qquad\qquad\qquad\qquad\qquad\quad 8x^3-\ 4x^2-16x-\ 4 \\
\qquad\qquad\qquad\qquad\qquad\quad \underline{8x^3-44x^2+44x+16} \\
\qquad\qquad\qquad\qquad\qquad\quad 20\,)\,\underline{40x^2-60x-20} \\
\qquad\qquad\qquad\qquad\qquad\qquad 2x^2-\ 3x-\ 1\,)\,2x^3-11x^2+11x+4\,(\,x-4 \\
\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\quad \underline{2x^3-\ 3x^2-\quad x} \\
\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad -\ 8x^2+12x+4 \\
\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad -\ 8x^2+12x+4
\end{array}
$$

Hence the H.C.F. is $2x^2-3x-1$.

32.

$$
\begin{array}{l}
2x^4+4x^3+3x^2-2x-2\,)\,3x^4+\ 6x^3+\ 7x^2+\ 2x+\ 2 \\
\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\quad \underline{\quad 2} \\
\qquad\qquad\qquad\qquad\qquad\quad 6x^4+12x^3+14x^2+\ 4x+\ 4\,(\,3 \\
\qquad\qquad\qquad\qquad\qquad\quad \underline{6x^4+12x^3+\ 9x^2-\ 6x-\ 6} \\
\qquad\qquad\qquad\qquad\qquad\qquad\qquad\quad 5\,)\,\underline{5x^2+10x+10} \\
\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad x^2+\ 2x+\ 2\,)\,2x^4+4x^3+3x^2-2x-2\,(\,2x^2-1 \\
\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \underline{2x^4+4x^3+4x^2} \\
\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad -\ x^2-2x-2 \\
\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad -\ x^2-2x-2
\end{array}
$$

Hence the H.C.F. is x^2+2x+2.

33.

$$x^4-x^3+2x^2-x-1\ )\ 2x^4-2x^2+x-1\ (\ 2$$
$$2x^4-2x^3+4x^2-2x-2$$
$$2x^3-6x^2+3x+1\ )\ 2x^4-2x^3+4x^2-2x-2\ (\ x+2$$
$$2x^4-6x^3+3x^2+x$$
$$4x^3+x^2-3x-2$$
$$4x^3-12x^2+6x+2$$
$$13x^2-9x-4$$

$$1+3x-6x^2+2x^3$$
$$16$$
$$4+9x-13x^2\ )\ 16+48x-96x^2+32x^3\ (\ 4+3x$$
$$16+36x-52x^2$$
$$12x-44x^2+32x^3$$
$$12x+27x^2-39x^3$$
$$-71x^2\ )\ -71x^2+71x^3$$
$$1-x\ )\ 4+9x-13x^2\ (\ 4+13x$$
$$4-4x$$
$$13x-13x^2$$
$$13x-13x^2$$

34.

$$2x^4-6x^3+3x^2-3x+1\ )\ x^7-3x^6+x^5-4x^2+12x-4$$
$$2$$
$$2x^7-6x^6+2x^5-8x^2+24x-8\ (\ x^3$$
$$2x^7-6x^6+3x^5-3x^4+x^3$$
$$-x^5+3x^4-x^3-8x^2+24x-8$$
$$2$$
$$-2x^5+6x^4-2x^3-16x^2+48x-16\ (\ -x$$
$$-2x^5+6x^4-3x^3+3x^2-x$$
$$x^3-19x^2+49x-16\ )\ 2x^4-6x^3+3x^2-3x+1\ (\ 2x+32$$
$$2x^4-38x^3+98x^2-32x$$
$$32x^3-95x^2+29x+1$$
$$32x^3-608x^2+1568x-512$$
$$513\ )\ 513x^2-1539x+513$$
$$x^2-3x+1$$

$$x^2-3x+1\ )\ x^3-19x^2+49x-16\ (\ x-16$$
$$x^3-3x^2+x$$
$$-16x^2+48x-16$$
$$-16x^2+48x-16$$

35.

$$x^3+(m-3)\,x^2-(2m^2+3m)\,x+6m^2\ )\ x^3+(5m-3)\,x^2+(6m^2-15m)\,x-18m^2\ (\ 1$$
$$x^3+\ (m-3)\,x^2-(2m^2+\ 3m)\,x+\ 6m^2$$
$$4m\ )\ 4mx^2+(8m^2-12m)\,x-24m^2$$
$$x^2+(2m-3)\,x-6m\ )\ x^3+(\ m-3)\,x^2-(2m^2+3m)\,x+6m^2\ (\ x-m$$
$$x^3+(2m-3)\,x^2-\qquad 6mx$$
$$-mx^2-(2m^2-3m)\,x+6m^2$$
$$-mx^2-(2m^2-3m)\,x+6m^2$$

36.

$$mnx^2+(m^2+n^2)\,xy+mny^2\ )\ mnx^3+n^2x^2y+m^2xy^2+mny^3\ (\ x$$
$$mnx^3+(m^2+n^2)\,x^2y+mnxy^2$$
$$-m^2x^2y+(m^2-mn)\,xy^2+mny^3$$
$$n$$
$$-m^2nx^2y+(m^2n-mn^2)\,xy^2+mn^2y^3\ (\ -my$$
$$-m^2nx^2y-(m^3\ +mn^2)\,xy^2-m^2ny^3$$
$$(m^2+mn)\,y^2\ )\ (m^3+m^2n)\,xy^2+(m^2n+mn^2)\,y^3$$
$$mx+ny\ )\ mnx^2+(m^2+n^2)\,xy+mny^2\ (\ nx+my$$
$$mnx^2+\qquad n^2\ xy$$
$$m^2xy\qquad +mny^2$$
$$m^2xy\qquad +mny^2$$

XXXIII. PAGE 118.

5. We have $(x+1)\,(x+2)$ and $(x+1)\,(x+4)$.
Hence L. C. M. is $(x+1)\,(x+2)\,(x+4)$.

6. We have $(x+y)\,(x+2y)$ and $(x+y)\,(x+4y)$.
Hence the L. C. M. is $(x+y)\,(x+2y)\,(x+4y)$.

7. We have $(x-1)\,(x-3)$ and $(x-2)\,(x-3)$.
Hence the L. C. M. is $(x-1)\,(x-2)\,(x-3)$.

8. We have $(3x-y^2)(x-y^2)$ and $(3x-y^2)(2x-y^2)$.
Hence the L. C. M. is $(3x-y^2)(2x-y^2)(x-y^2)$.

11. We have $(x+3)(x+4)$, $(x+2)(x+4)$ and $(x+2)(x+3)$.
Hence the L. C. M. is $(x+2)(x+3)(x+4)$.

12. We have $(x-3y)(x-4y)$, $(x-2y)(x-4y)$ and $(x-2y)(x-3y)$.
Hence the L. C. M. is $(x-2y)(x-3y)(x-4y)$.

XXXIV. Pages 119, 120.

6. We have $a^2(a+b)$, $a(a-b)$ and $(a-b)(a+b)$.
Hence the L. C. M. is $a^2(a+b)(a-b)$.

7. We have $y^2(x-1)$, $xy^2(x+1)$ and $y(x-1)(x+1)$.
Hence the L. C. M. is $xy^2(x-1)(x+1)$.

9. We have $6(x-3)(x+3)$, $9(x+3)$, $15(x-4)$ and $10(x+3)(x-4)$.
Hence the L. C. M. is $90(x-3)(x+3)(x-4)$.

10. We have $4(a+b)$, $6(a-2b)(a+2b)$, $(a-b)(a-2b)$.
Hence the L. C. M. is $12(a+b)(a-b)(a+2b)(a-2b)$.

11. We have $4b^3(a+d)$, $b^2(a-d)(a+d)$, $8abd(a-d)$.
Hence the L. C. M. is $8ab^3d(a-d)(a+d)$.

12. We have $(x-4)(x-6)$, $(x-2)(x-6)$ and $(x-2)(x-4)$.
Hence the L. C. M. is $(x-2)(x-4)(x-6)$.

13. We have $(x+1)(x-10)$, $(x+3)(x-10)$ and $(x+1)(x+3)$.
Hence the L. C. M. is $(x+1)(x+3)(x-10)$.

14. We have $2(x-2)(x+2)$, $3(x-1)(x-2)$ and $6(x+1)(x+2)$.
Hence the L. C. M. is $6(x-1)(x+1)(x-2)(x+2)$.

15. We have $(x-1)(x-2)$, $(2x+3)(x-2)$ and $(3x+1)(x-1)$.
Hence the L. C. M. is $(x-1)(x-2)(2x+3)(3x+1)$.

16. We have $(2x-1)(3x+2)$, $(7x+1)(3x+2)$, and $(7x+1)(2x-1)$, &c.

17. We have $(3x-y)(x-3y)$, $(3x-y)(x-y)$ and $(x-y)(x-3y)$, &c.

18. We have $x(x-1)(x+3)$ and $x(2x-1)(x+3)$, &c.

19. We have $y^2(x-3)(x+3)$, $y(x-3)(x+2)$ and $x(x+3)(x-2)$, &c.

20. We have $(x-a)(x^2+ax+a^2)$, $(x+a)(x^2-ax+a^2)$ and
$(x^2-ax+a^2)(x^2+ax+a^2)$, &c.

21. The expressions are $(x-1)(x+1)$, $(x+1)(x^2+1)$ and $(x-1)(x^2+1)$, &c.

22. The H. C. F. will be found to be $3x^2+2x+1$.
And the expressions are $(3x^2+2x+1)(x-4)$, and $(3x^2+2x+1)(3x-2)$.

23. $x^3-ax^2-a^2x+a^3=x^2(x-a)-a^2(x-a)=(x^2-a^2)(x-a).$
$x^3+ax^2-a^2x-a^3=x^2(x+a)-a^2(x+a)=(x^2-a^2)(x+a).$
Hence the L. C. M. is $(x^2-a^2)(x-a)(x+a)=(x^2-a^2)^2$.

24. The H. C. F. will be found to be x^2+3x+2.
Then we have $(x^2+3x+2)(x-2)$ and $(x^2+3x+2)(x+3)$, &c.

25. The H. C. F. will be found to be x^2-2x+4.
Then we have $(x^2-2x+4)(x+6)$ and $(x^2-2x+4)(x^2+x-2)$, &c.

26. The H. C. F. will be found to be $a+1$.
Then $a^3+6a^2+11a+6=(a+1)(a^2+5a+6)$,
and $a^3+10a^2+29a+20=(a+1)(a^2+9a+20)$, &c.

XXXVI. PAGES 126, 127.

1. $\dfrac{2ab}{a(a+b)}=\dfrac{2b}{(a+b)}.$

2. $\dfrac{x^2y^2}{x^2(1-y^2)}=\dfrac{y^2}{1-y^2}.$

3. $\dfrac{a(a-b)}{a(a+b)}=\dfrac{a-b}{a+b}.$

4. $\dfrac{x(x+a)}{(x-a)(x+a)}=\dfrac{x}{x-a}.$

5. $\dfrac{(x-1)^2}{(x-1)(x+1)}=\dfrac{x-1}{x+1}.$

6. $\dfrac{(x-xy)(x+xy)}{(x+xy)^2}=\dfrac{x-xy}{x+xy}=\dfrac{x(1-y)}{x(1+y)}.$

7. $\dfrac{x(x+2)}{(x-2)(x+2)}=\dfrac{x}{x-2}.$

8. $\dfrac{x^2(x^2+1)}{(x^2-1)(x^2+1)}=\dfrac{x^2}{x^2-1}.$

9. $\dfrac{4(x-4)}{(x-4)(x+4)}=\dfrac{4}{x+4}.$

10. $\dfrac{2x^3(1-2x)}{x^2(1-2x)(1+2x)}=\dfrac{2x}{1+2x}.$

11. $\dfrac{x-2}{(2-x)(2+x)}=-\dfrac{1}{2+x}.$ (Cf. Ex. 4, Art. 130.)

12. $\dfrac{a-3}{(3-a)(3+a)}=-\dfrac{1}{3+a}.$

13. $\dfrac{x^2(a^2-x^2)}{(x^2-a^2)(x^2+a^2)}=-\dfrac{x^2}{x^2+a^2}.$

14. $\dfrac{(x^2-a)(x^2+a)}{a(a-x^2)}=-\dfrac{x^2+a}{a}.$

15. $\dfrac{x^3(x^2-a^2)}{(x^2-a^2)(x^2+a^2)}=\dfrac{x^3}{x^2+a^2}.$

16. $\dfrac{x^2y^2(a^2-x^2y^2)}{(x^2y^2-a^2)(x^2y^2+a^2)}=-\dfrac{x^2y^2}{x^2y^2+a^2}.$

17. $\dfrac{5a(x-3a)}{(x-3a)(x+3a)}=\dfrac{5a}{x+3a}.$

18. $\dfrac{3x(x-4a)}{3(16a^2-x^2)}=\dfrac{3x(x-4a)}{3(4a-x)(4a+x)}=-\dfrac{x}{4a+x}.$

19. $\dfrac{(a-x)^2}{(a-x)(a+x)}=\dfrac{a-x}{a+x}.$

20. $\dfrac{(a^2+b^2)^2}{(a^2-b^2)(a^2+b^2)}=\dfrac{a^2+b^2}{a^2-b^2}.$

21. $\dfrac{(x-1)(x+1)}{(x-1)(x^2+x+1)}=\dfrac{x+1}{x^2+x+1}.$

22. $\dfrac{(x^2-1)(x^2+1)}{(x^2-1)(x^4+x^2+1)}=\dfrac{x^2+1}{x^4+x^2+1}.$

23. $\frac{(x-2)(x-3)}{(x-3)(x-4)}=\frac{x-2}{x-4}$.

24. $\frac{(1-2a)(1-3a)}{(1-3a)(1-4a)}=\frac{1-2a}{1-4a}$.

25. $\frac{(x-4)(x-5)}{(x+11)(x-5)}=\frac{x-4}{x+11}$.

26. $\frac{(1-4y^2)(1-5y^2)}{(1+11y^2)(1-5y^2)}=\frac{1-4y^2}{1+11y^2}$.

27. $\frac{(x-y)(x-7y)}{(x+4y)(x-7y)}=\frac{x-y}{x+4y}$.

28. $\frac{(1-a^2b^2)(1-7a^2b^2)}{(1+4a^2b^2)(1-7a^2b^2)}=\frac{1-a^2b^2}{1+4a^2b^2}$.

29. $\frac{(a-x)(a^2+ax+x^2)(a+x)}{(a+x)(a^2-ax+x^2)(a-x)}=\frac{a^2+ax+x^2}{a^2-ax+x^2}$.

30. $\frac{(a-b)(a^2+ab+b^2)(a^2-ab+b^2)}{(a+b)(a^2-ab+b^2)(a^2+ab+b^2)}=\frac{a-b}{a+b}$.

31. $\frac{(a^2-b^2)(a^2+b^2)}{(a-b)(a^2+ab+b^2)(a+b)}=\frac{(a-b)(a+b)(a^2+b^2)}{(a-b)(a^2+ab+b^2)(a+b)}=\frac{a^2+b^2}{a^2+ab+b^2}$.

32. $\frac{(a^2-b^2)(a^4+a^2b^2+b^4)(a-b)}{(a-b)(a^2+ab+b^2)(a^2-b^2)(a^2+b^2)}=\frac{a^4+a^2b^2+b^4}{(a^2+ab+b^2)(a^2+b^2)}$

$$=\frac{(a^2-ab+b^2)(a^2+ab+b^2)}{(a^2+ab+b^2)(a^2+b^2)}=\frac{a^2-ab+b^2}{a^2+b^2}.$$

XXXVII. PAGES 127, 128.

1. The H.C.F. of the numerator and denominator will be found to be x^2-2x+1.

Then $$\frac{x^3-3x+2}{2x^3-3x^2+1}=\frac{(x^2-2x+1)(x+2)}{(x^2-2x+1)(2x+1)}=\&c.$$

2. The H.C.F. will be found to be $x-1$; then

$$\frac{3x^2-8x+5}{x^3-4x^2+5x-2}=\frac{(x-1)(3x-5)}{(x-1)(x^2-3x+2)}=\&c.$$

3.
$$\begin{array}{r}
x^3+3x^2-20\ )\ x^4-\ \ x^2-12\ \ (x-3 \\
x^4+3x^3-20x \\ \hline
-3x^3-\ \ x^2+20x-12 \\
-3x^3-\ 9x^2\ \ \ \ \ \ +60 \\ \hline
4\ )\ 8x^2+20x-72 \\ \hline
2x^2+\ 5x-18\ )\ x^3+3x^2-20 \\
4 \\ \hline
4x^3+12x^2-80\ \ (\ 2x+1 \\
4x^3+10x^2-36x \\ \hline
2x^2+36x-80 \\
2x^2+\ 5x-18 \\ \hline
31\ )\ 31x-62 \\ \hline
x-\ 2\ )\ 2x^2+5x-18\ (\ 2x+9 \\
2x^2-4x \\ \hline
9x-18 \\
9x-18
\end{array}$$

Then $\frac{x^3+3x^2-20}{x^4-x^2-12}=\frac{(x-2)(x^2+5x+10)}{(x-2)(x^3+2x^2+3x+6)}=\&c.$

4. The H.C.F. will be found to be x^2+5x+6. Then

$$\frac{x^3+6x^2+11x+6}{x^3+5x^2+6x}=\frac{(x^2+5x+6)(x+1)}{(x^2+5x+6)x}=\&c.$$

5.

$$\begin{array}{l}
x^3-7ax^2+8a^2x-2a^3\ )\ 2x^3+\ ax^2+\ 4a^2x-7a^3\ (\ 2 \\
\qquad\qquad\qquad\qquad\quad \underline{2x^3-14ax^2+16a^2x-4a^3} \\
\qquad\qquad\qquad\qquad\quad 3a\ )\ \underline{15ax^2-12a^2x-3a^3} \\
\qquad\qquad\qquad\qquad\qquad\quad 5x^2-\ 4ax\ -\ a^2\ )\ x^3-\ 7ax^2+\ 8a^2x-2a^3\ (\ 2a-16x \\
\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\quad \underline{10ax^2-\ 8a^2x-2a^3} \\
\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\quad x^3-17ax^2+16a^2x \\
\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\ \underline{-80x^3+64ax^2+16a^2x} \\
\qquad\qquad\qquad\qquad\qquad\qquad\qquad 81x^2\ )\ \underline{81x^3-81ax^2} \\
\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad x\ -\ a\ )\ 5x^2-4ax-a^2\ (\ 5x+a \\
\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \underline{5x^2-5ax} \\
\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad ax-a^2 \\
\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad ax-a^2
\end{array}$$

Then $$\frac{2x^3+ax^2+4a^2x-7a^3}{x^3-7ax^2+8a^2x-2a^3}=\frac{(x-a)(2x^2+3ax+7a^2)}{(x-a)(x^2-6ax+2a^2)}=\&c.$$

6. The H.C.F. will be found to be $2x-1$.

Then $$\frac{2x^3+3x^2+4x-3}{6x^3+x^2-1}=\frac{(2x-1)(x^2+2x+3)}{(2x-1)(3x^2+2x+1)}=\&c.$$

7. The H.C.F. will be found to be x^2+2x+1.

Then $$\frac{x^3-3x-2}{x^4+2x^3+2x^2+2x+1}=\frac{(x^2+2x+1)(x-2)}{(x^2+2x+1)(x^2+1)}=\&c.$$

8. The H.C.F. will be found to be x^2+x+1.

Then $$\frac{x^4-x^3-x+1}{x^4-2x^3-x^2-2x+1}=\frac{(x^2+x+1)(x^2-2x+1)}{(x^2+x+1)(x^2-3x+1)}=\&c.$$

9. The H.C.F. will be found to be x^3-3x-1.

Then $$\frac{x^4+2x^3-3x^2-7x-2}{2x^4+x^3-6x^2-5x-1}=\frac{(x^3-3x-1)(x+2)}{(x^3-3x-1)(2x+1)}=\&c.$$

10.
$$\begin{array}{l}
x^4-20x^2-15x+4\;)\;x^4+9x^3+19x^2-9x-20\;(\;1\\
\qquad\qquad\qquad\qquad\;\; x^4\qquad\;\; -20x^2-15x+4\\
\qquad\qquad\qquad\qquad\;\; \overline{3\;)\;9x^3+39x^2+6x-24}\\
\qquad\qquad\qquad\qquad\qquad \overline{3x^3+13x^2+2x-8}\;)\;x^4-20x^2-15x+4\\
\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\quad 3\\
\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \overline{3x^4-60x^2-45x+12}\;(\;x\\
\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad 3x^4+13x^3+2x^2-8x\\
\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \overline{\quad-13x^3-62x^2-37x+12}\\
\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\quad 3\\
\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\quad \overline{-39x^3-186x^2-111x+36}\;(\;-13\\
\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\quad -39x^3-169x^2-26x+104\\
\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\quad \overline{-17\;)\;-17x^2-85x-68}\\
\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\quad \overline{x^2+5x+4}\;)\;3x^3+13x^2+2x-8\;(\;3x-2\\
\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\quad 3x^3+15x^2+12x\\
\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\quad \overline{\quad-2x^2-10x-8}\\
\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad -2x^2-10x-8
\end{array}$$

Then $$\frac{x^4-20x^2-15x+4}{x^4+9x^3+19x^2-9x-20}=\frac{(x^2+5x+4)(x^2-5x+1)}{(x^2+5x+4)(x^2+4x-5)}=\&c.$$

XXXVIII. PAGES 132—134.

1. L. C. M. of denominators is $30x$.

Then $$\frac{2}{3x}=\frac{2\times 10}{3x\times 10}=\frac{20}{30x},\quad \frac{4}{5x}=\frac{4\times 6}{5x\times 6}=\frac{24}{30x}.$$

2. L. C. M. of denominators is $24abcx$.

Then $$\frac{1}{2ax}=\frac{1\times 12bc}{2ax\times 12bc}=\frac{12bc}{24abcx};\quad \frac{1}{6bx}=\frac{1\times 4ac}{6bx\times 4ac}=\frac{4ac}{24abcx};\quad \frac{1}{8cx}=\frac{1\times 3ab}{8cx\times 3ab}=\frac{3ab}{24abcx}.$$

3. L. C. M. of denominators is abc.

Then $\dfrac{a}{bc}=\dfrac{a\times a}{bc\times a}=\dfrac{a^2}{abc}$; $\dfrac{b}{ca}=\dfrac{b\times b}{ca\times b}=\dfrac{b^2}{abc}$; $\dfrac{c}{ab}=\dfrac{c\times c}{ab\times c}=\dfrac{c^2}{abc}$.

4. L. C. M. is abc.

Then $\dfrac{bc}{a}=\dfrac{bc\times bc}{a\times bc}=\dfrac{b^2c^2}{abc}$; $\dfrac{ca}{b}=\dfrac{ca\times ac}{b\times ac}=\dfrac{c^2a^2}{abc}$; $\dfrac{ab}{c}=\dfrac{ab\times ab}{c\times ab}=\dfrac{a^2b^2}{abc}$.

5. The L. C. M. is $2(x+1)$.

6. L. C. M. is $6(x-1)$.

Then $\dfrac{3}{2(x-1)}=\dfrac{3\times 3}{2(x-1)\times 3}=\dfrac{9}{6(x-1)}$; $\dfrac{4}{3(x-1)}=\dfrac{4\times 2}{3(x-1)\times 2}=\dfrac{8}{6(x-1)}$.

7. L. C. M. is $24(x+2)$.

Then
$$\frac{5}{6(x+2)}=\frac{5\times 4}{6(x+2)\times 4}=\frac{20}{24(x+2)};$$
$$\frac{3}{8(x+2)}=\frac{3\times 3}{8(x+2)\times 3}=\frac{9}{24(x+2)}.$$

8. L. C. M. of $x+1$, $2(x+1)$, and $(x-1)(x+1)$ is $2(x-1)(x+1)$.

Then
$$\frac{1}{x+1}=\frac{1\times 2(x-1)}{(x+1)\times 2(x-1)}=\frac{2(x-1)}{2(x^2-1)};$$
$$\frac{3}{2(x+1)}=\frac{3\times(x-1)}{2(x+1)\times(x-1)}=\frac{3(x-1)}{2(x^2-1)};$$
$$\frac{5}{(x-1)(x+1)}=\frac{5\times 2}{(x-1)(x+1)\times 2}=\frac{10}{2(x^2-1)}.$$

9. L. C. M. of $5(x-1)$, $3(x+1)$, and $(x-1)(x+1)$ is $15(x-1)(x+1)$.
$$\frac{6}{5(x-1)}=\frac{6\times 3(x+1)}{5(x-1)\times 3(x+1)}=\frac{18(x+1)}{15(x^2-1)};$$
$$\frac{2}{3(x+1)}=\frac{2\times 5(x-1)}{3(x+1)\times 5(x-1)}=\frac{10(x-1)}{15(x^2-1)};$$
$$\frac{4}{(x-1)(x+1)}=\frac{4\times 15}{(x-1)(x+1)\times 15}=\frac{60}{15(x^2-1)}.$$

10. The fractions are $\dfrac{a}{x-a}$, $\dfrac{-x}{x-a}$, $\dfrac{a^2}{x^2-a^2}$, $\dfrac{-x^2}{x^2-a^2}$.

[See Ex. 2, Art. 133.]

L. C. M. is $(x-a)(x+a)$.
$$\frac{a}{x-a}=\frac{a\times(x+a)}{(x-a)\times(x+a)}=\frac{a(x+a)}{x^2-a^2};$$
$$\frac{-x}{(x-a)}=\frac{-x\times(x+a)}{(x-a)\times(x+a)}=\frac{-x(x+a)}{x^2-a^2}.$$

11. The fractions are $\dfrac{2a}{a-b}$, $\dfrac{-b}{2a-2b}$, $\dfrac{3a^2}{4(a^2-b^2)}$, $\dfrac{-5b^2}{6(a^2-b^2)}$.

L. C. M. of $(a-b)$, $2(a-b)$, $4(a-b)(a+b)$, and $6(a-b)(a+b)$ is

$$12(a-b)(a+b).$$

Then
$$\frac{2a}{a-b}=\frac{2a\times 12(a+b)}{(a-b)\times 12(a+b)}=\frac{24a(a+b)}{12(a^2-b^2)};$$
$$\frac{-b}{2(a-b)}=\frac{-b\times 6(a+b)}{2(a-b)\times 6(a+b)}=\frac{-6b(a+b)}{12(a^2-b^2)};$$
$$\frac{3a^2}{4(a-b)(a+b)}=\frac{3a^2\times 3}{4(a-b)(a+b)\times 3}=\frac{9a^2}{12(a^2-b^2)};$$
$$\frac{-5b^2}{6(a-b)(a+b)}=\frac{-5b^2\times 2}{6(a-b)(a+b)\times 2}=\frac{-10b^2}{12(a^2-b^2)}.$$

12. L. C. M. is $(x+1)^3$. Then $\dfrac{1}{x+1}=\dfrac{1\times(x+1)^2}{(x+1)\times(x+1)^2}=\dfrac{(x+1)^2}{(x+1)^3}$;
$$\frac{2x}{(x+1)^2}=\frac{2x\times(x+1)}{(x+1)^2\times(x+1)}=\frac{2x(x+1)}{(x+1)^3}.$$

13. $\dfrac{1}{(b-x)(c-x)}=\dfrac{1}{\{-(x-b)\}\{-(x-c)\}}=\dfrac{1}{(x-b)(x-c)}$. Thus we have
$$\frac{1}{(x-a)(x-b)},\quad \frac{1}{(x-b)(x-c)},\quad \frac{1}{(x-c)(x-a)}.$$
L. C. M. of denominators is $(x-a)(x-b)(x-c)$.

Then
$$\frac{1}{(x-a)(x-b)}=\frac{x-c}{(x-a)(x-b)(x-c)};$$
$$\frac{1}{(x-b)(\text{[illegible]}-c)}=\frac{x-a}{(x-a)(x-b)(x-c)};\quad \frac{1}{(x-c)(x-a)}=\frac{x-b}{(x-a)(x-b)(x-c)}.$$

14.
$$\frac{1}{(b-c)(b-a)}=\frac{1}{(b-c)\{-(a-b)\}}=\frac{-1}{(b-c)(a-b)};$$
also
$$\frac{1}{(c-a)(c-b)}=\frac{1}{\{-(a-c)\}\{-(b-c)\}}=\frac{1}{(a-c)(b-c)}.$$
Thus we have $\dfrac{1}{(a-b)(a-c)}$, $\dfrac{-1}{(b-c)(a-b)}$, $\dfrac{1}{(a-c)(b-c)}$.

L. C. M. is $(a-b)(a-c)(b-c)$; $\dfrac{1}{(a-b)(a-c)}=\dfrac{(b-c)}{(a-b)(a-c)(b-c)}$;
$$\frac{-1}{(b-c)(a-b)}=\frac{-(a-c)}{(a-b)(a-c)(b-c)};\quad \frac{1}{(a-c)(b-c)}=\frac{(a-b)}{(a-b)(a-c)(b-c)}.$$

15. $$\frac{a-1}{1}+\frac{2}{a+1}=\frac{(a-1)(a+1)}{a+1}+\frac{2}{a+1}=\frac{(a-1)(a+1)+2}{a+1}=\frac{a^2+1}{a+1}.$$

16. $$\frac{a+x}{1}+\frac{x^2}{a-x}=\frac{(a+x)(a-x)}{a-x}+\frac{x^2}{a-x}=\frac{(a-x)(a+x)+x^2}{a-x}=\frac{a^2}{a-x}.$$

17. $$\frac{x+2y}{1}+\frac{4y^2}{x-2y}=\frac{(x+2y)(x-2y)}{x-2y}+\frac{4y^2}{x-2y}=\frac{(x-2y)(x+2y)+4y^2}{x-2y}$$
$$=\frac{x^2}{x-2y}.$$

18. $$\frac{a-b}{a^2b}+\frac{a-b}{ab^2}=\frac{(a-b)\,b}{a^2b^2}+\frac{(a-b)\,a}{a^2b^2}=\frac{(a-b)\,b+(a-b)\,a}{a^2b^2}=\frac{a^2-b^2}{a^2b^2}.$$

19. $$\frac{(a-5b)\times 3}{5\times 3}-\frac{(a-3b)\times 5}{3\times 5}=\frac{3\,(a-5b)-5\,(a-3b)}{3\times 5}=\frac{-2a}{15}.$$

20. $$\frac{(6a-5b)\times 2}{3\times 2}-\frac{(4a-7b)\times 3}{2\times 3}=\frac{2\,(6a-5b)-3\,(4a-7b)}{2\times 3}=\frac{11b}{6}.$$

21. $$\frac{5\,(a-3b)}{4\times 5}+\frac{4\,(3a-b)}{5\times 4}=\frac{5\,(a-3b)+4\,(3a-b)}{5\times 4}=\frac{17a-19b}{20}.$$

22. $$\frac{2\,(x-y)}{3}-\frac{3\,(x+y)}{4}=\frac{2\,(x-y)\times 4}{3\times 4}-\frac{3\,(x+y)\times 3}{4\times 3}=\frac{8\,(x-y)-9\,(x+y)}{4\times 3}$$
$$=\frac{-x-17y}{12}=-\frac{x+17y}{12}.$$

23. $$\frac{5\,(x-2y)}{6}-\frac{3\,(y-2x)}{4}=\frac{10\,(x-2y)}{12}-\frac{9\,(y-2x)}{12}$$
$$=\frac{10\,(x-2y)-9\,(y-2x)}{12}=\frac{28x-29y}{12}.$$

24. $$\frac{3x}{12}-\frac{4\,(x-4)}{12}+\frac{2\,(x-5)}{12}=\frac{3x-4\,(x-4)+2\,(x-5)}{12}=\frac{x+6}{12}.$$

25. $$\frac{4\,(2x-3y)}{12}+\frac{3\,(x+2y)}{12}-\frac{2\,(3x-2y)}{12}$$
$$=\frac{4\,(2x-3y)+3\,(x+2y)-2\,(3x-2y)}{12}=\frac{5x-2y}{12}.$$

26. $$\frac{a}{a-b}+\frac{-b}{a-b}=\frac{a-b}{a-b}=1.$$

27. $$\frac{x}{x-a}+\frac{-a}{x-a}=\frac{x-a}{x-a}=1.$$

28. $$\frac{x}{x^2-a^2}+\frac{-a}{x^2-a^2}=\frac{x-a}{x^2-a^2}=\frac{x-a}{(x-a)\,(x+a)}=\frac{1}{x+a}.$$

29. $$\frac{1+x}{(1-x)\,(1+x)}-\frac{2}{(1-x)\,(1+x)}=\frac{1+x-2}{(1-x)\,(1+x)}=\frac{x-1}{(1-x)\,(1+x)}$$
$$=\frac{-(1-x)}{(1-x)\,(1+x)}=-\frac{1}{1+x}.$$

30. $$\frac{2+x}{(2-x)\,(2+x)}-\frac{4}{(2-x)\,(2+x)}=\frac{2+x-4}{(2-x)\,(2+x)}=\frac{x-2}{(2-x)\,(2+x)}$$
$$=\frac{-(2-x)}{(2-x)\,(2+x)}=-\frac{1}{2+x}.$$

31. $$\frac{(x-3)}{(3+x)(x-3)}+\frac{6}{(x+3)(x-3)}=\frac{x-3+6}{(x+3)(x-3)}=\frac{x+3}{(x+3)(x-3)}=\frac{1}{x-3}.$$

32. $$\frac{x-2}{(x-3)(x-2)}-\frac{x-3}{(x-2)(x-3)}=\frac{(x-2)-(x-3)}{(x-3)(x-2)}=\frac{1}{(x-3)(x-2)}.$$

33. $$\frac{x-5y}{(x-4y)(x-5y)}-\frac{x-4y}{(x-5y)(x-4y)}=\frac{(x-5y)-(x-4y)}{(x-4y)(x-5y)}$$
$$=\frac{-y}{(x-4y)(x-5y)}.$$

34. $$\frac{5x-2y}{(3x-2y)(5x-2y)}-\frac{3x-2y}{(5x-2y)(3x-2y)}=\frac{(5x-2y)-(3x-2y)}{(3x-2y)(5x-2y)}$$
$$=\frac{2x}{(3x-2y)(5x-2y)}.$$

35. $$\frac{(1+x)^2}{(1-x)(1+x)}-\frac{(1-x)^2}{(1+x)(1-x)}=\frac{(1+x)^2-(1-x)^2}{(1-x)(1+x)}$$
$$=\frac{(1+2x+x^2)-(1-2x+x^2)}{1-x^2}=\frac{4x}{1-x^2}.$$

36. $$\frac{(a+2b)^2}{(a-2b)(a+2b)}-\frac{(a-2b)^2}{(a+2b)(a-2b)}=\frac{(a+2b)^2-(a-2b)^2}{(a-2b)(a+2b)}$$
$$=\frac{(a^2+4ab+4b^2)-(a^2-4ab+4b^2)}{a^2-4b^2}=\frac{8ab}{a^2-4b^2}.$$

37. $$\frac{x-2}{(x-2)^2}+\frac{2}{(x-2)^2}=\frac{x-2+2}{(x-2)^2}=\frac{x}{(x-2)^2}.$$

38. $$\frac{a(a-2b)}{(a-2b)^2}+\frac{2ab}{(a-2b)^2}=\frac{a(a-2b)+2ab}{(a-2b)^2}=\frac{a^2}{(a-2b)^2}.$$

39. $$\frac{(x+y)^2}{(x-y)(x+y)}-\frac{(x-y)^2}{(x+y)(x-y)}+\frac{4y^2}{(x-y)(x+y)}$$
$$=\frac{(x+y)^2-(x-y)^2+4y^2}{(x-y)(x+y)}=\frac{4xy+4y^2}{(x-y)(x+y)}=\frac{4y(x+y)}{(x-y)(x+y)}=\&c.$$

40. $$\frac{2a(a-b)}{(a+b)(a-b)}+\frac{2b(a+b)}{(a-b)(a+b)}-\frac{a^2+b^2}{(a-b)(a+b)}$$
$$=\frac{2a(a-b)+2b(a+b)-(a^2+b^2)}{(a-b)(a+b)}=\frac{a^2+b^2}{a^2-b^2}.$$

41. $$\frac{a^2}{(a-1)a}-\frac{a(a-1)}{a(a-1)}-\frac{1}{a(a-1)}=\frac{a^2-a(a-1)-1}{a(a-1)}=\frac{a-1}{a(a-1)}=\frac{1}{a}.$$

42. $$\frac{2(x+2)}{(x-2)(x+2)}-\frac{(x-2)}{(x+2)(x-2)}-\frac{x+6}{x^2+4}=\frac{2(x+2)-(x-2)}{(x-2)(x+2)}-\frac{x+6}{x^2+4}$$
$$=\frac{x+6}{x^2-4}-\frac{x+6}{x^2+4}=\frac{(x+6)(x^2+4)-(x+6)(x^2-4)}{(x^2-4)(x^2+4)}$$
$$=\frac{(x+6)\{(x^2+4)-(x^2-4)\}}{x^4-16}=\frac{8(x+6)}{x^4-16}=\&c.$$

43.
$$\frac{2(x+1)}{(x-1)\,2(x+1)}-\frac{x-1}{2(x+1)(x-1)}-\frac{x+3}{2(x^2+1)}$$
$$=\frac{2(x+1)-(x-1)}{2(x+1)(x-1)}-\frac{x+3}{2(x^2+1)}=\frac{x+3}{2(x^2-1)}-\frac{x+3}{2(x^2+1)}$$
$$=\frac{(x+3)(x^2+1)-(x+3)(x^2-1)}{2(x^2-1)(x^2+1)}=\frac{(x+3)\{(x^2+1)-(x^2-1)\}}{2(x^4-1)}$$
$$=\frac{2(x+3)}{2(x^4-1)}=\&c.$$

44.
$$\frac{(3-x)(1+3x)-(3+x)(1-3x)}{(1+3x)(1-3x)}-\frac{1-16x}{9x^2-1}$$
$$=\frac{(3-x)(1+3x)-(3+x)(1-3x)-(16x-1)}{1-9x^2}$$
$$=\frac{(3+8x-3x^2)-(3-8x-3x^2)-(16x-1)}{1-9x^2}=\frac{1}{1-9x^2}.$$

45.
$$\frac{a(a+x)}{(a-x)(a+x)}+\frac{a(a-x)}{(a+x)(a-x)}+\frac{2a^2}{a^2+x^2}=\frac{a(a+x)+a(a-x)}{(a-x)(a+x)}+\frac{2a^2}{a^2+x^2}$$
$$=\frac{2a^2}{a^2-x^2}+\frac{2a^2}{a^2+x^2}=\frac{2a^2(a^2+x^2)+2a^2(a^2-x^2)}{(a^2-x^2)(a^2+x^2)}=\frac{4a^4}{a^4-x^4}.$$

46.
$$\frac{a}{a-x}+\frac{a}{a+x}+\frac{2a^2}{a^2+x^2}+\frac{4a^4}{a^4+x^4}=\frac{2a^2}{a^2-x^2}+\frac{2a^2}{a^2+x^2}+\frac{4a^4}{a^4+x^4}$$
$$=\frac{4a^4}{a^4-x^4}+\frac{4a^4}{a^4+x^4}=\frac{4a^4(a^4+x^4)+4a^4(a^4-x^4)}{(a^4-x^4)(a^4+x^4)}=\frac{8a^8}{a^8-x^8}.$$

47.
$$\frac{(x+2)-(x+1)}{(x+1)(x+2)}+\frac{(x+4)-(x+3)}{(x+3)(x+4)}=\frac{1}{(x+1)(x+2)}+\frac{1}{(x+3)(x+4)}$$
$$=\frac{(x+3)(x+4)+(x+1)(x+2)}{(x+1)(x+2)(x+3)(x+4)}=\frac{(x^2+7x+12)+(x^2+3x+2)}{(x+1)(x+2)(x+3)(x+4)}=\&c.$$

48.
$$\frac{x(x+1)-2(x-1)(x+1)+x(x-1)}{x(x-1)(x+1)}=\frac{2}{x(x^2-1)}.$$

49.
$$\frac{(a+1)(a+2)-2a(a+2)+a(a+1)}{a(a+1)(a+2)}=\frac{2}{a(a+1)(a+2)}.$$

50. Take the first and last terms together, and also the two middle terms. Thus
$$\frac{(x+3)-(x-3)}{(x-3)(x+3)}+\frac{-3(x+1)+3(x-1)}{(x-1)(x+1)}=\frac{6}{x^2-9}+\frac{-6}{x^2-1}$$
$$=\frac{6(x^2-1)-6(x^2-9)}{(x^2-1)(x^2-9)}=\&c.$$

51.
$$\frac{(a+3)-a}{a(a+3)}+\frac{-3(a+2)+3(a+1)}{(a+1)(a+2)}=\frac{3}{a(a+3)}+\frac{-3}{(a+1)(a+2)}$$
$$=\frac{3(a+1)(a+2)-3a(a+3)}{a(a+1)(a+2)(a+3)}=\&c.$$

52. Group the terms with the same coefficients. Thus

$$\frac{(x+2)+(x-2)}{(x-2)(x+2)}+\frac{-4(x+1)-4(x-1)}{(x-1)(x+1)}+\frac{6}{x}=\frac{2x}{x^2-4}+\frac{-8x}{x^2-1}+\frac{6}{x}$$

$$=\frac{2x(x^2-1)x-8x(x^2-4)x+6(x^2-4)(x^2-1)}{x(x^2-1)(x^2-4)}$$

$$=\frac{(2x^4-2x^2)-(8x^4-32x^2)+(6x^4-30x^2+24)}{x(x^2-1)(x^2-4)}=\frac{24}{x(x^2-1)(x^2-4)}.$$

53.

$$\frac{\begin{array}{r}(a+1)(a+2)(a+3)(a+4)-4a(a+2)(a+3)(a+4)+6a(a+1)(a+3)\\(a+4)-4a(a+1)(a+2)(a+4)+a(a+1)(a+2)(a+3)\end{array}}{a(a+1)(a+2)(a+3)(a+4)}$$

$$=\frac{\begin{array}{r}(a^4+10a^3+35a^2+50a+24)-(4a^4+36a^3+104a^2+96a)+(6a^4+48a^3+114a^2\\+72a)-(4a^4+28a^3+56a^2+32a)+(a^4+6a^3+11a^2+6a)\end{array}}{a(a+1)(a+2)(a+3)(a+4)}$$

$$=\frac{24}{a(a+1)(a+2)(a+3)(a+4)}.$$

Or thus: given expression

$$=\left(\frac{1}{a}-\frac{3}{a+1}+\frac{3}{a+2}-\frac{1}{a+3}\right)-\left(\frac{1}{a+1}-\frac{3}{a+2}+\frac{3}{a+3}-\frac{1}{a+4}\right).$$

The expression in the first bracket $=\dfrac{a+3-a}{a(a+3)}-\dfrac{3(a+2)-3(a+1)}{(a+1)(a+2)}$

$$=\frac{3}{a(a+3)}-\frac{3}{(a+1)(a+2)}=\frac{3(a+1)(a+2)-3a(a+3)}{a(a+1)(a+2)(a+3)}=\frac{6}{a(a+1)(a+2)(a+3)}.$$

Change a into $a+1$, and we get the value of the expression in the second bracket. Thus the whole expression

$$=\frac{6}{a(a+1)(a+2)(a+3)}-\frac{6}{(a+1)(a+2)(a+3)(a+4)}$$

$$=\frac{24}{a(a+1)(a+2)(a+3)(a+4)}.$$

54.

$$\frac{1}{(x-2)(x-3)}-\frac{2}{(x-1)(x-3)}+\frac{1}{(x-1)(x-2)}$$

$$=\frac{x-1}{(x-2)(x-3)(x-1)}-\frac{2(x-2)}{(x-1)(x-3)(x-2)}+\frac{x-3}{(x-1)(x-2)(x-3)}$$

$$=\frac{(x-1)-2(x-2)+(x-3)}{(x-1)(x-2)(x-3)}=\frac{0}{(x-1)(x-2)(x-3)}=0.$$

55.

$$\frac{1}{(x+2a)(x+3a)}-\frac{2}{(x+a)(x+3a)}+\frac{1}{(x+a)(x+2a)}$$

$$=\frac{x+a}{(x+2a)(x+3a)(x+a)}-\frac{2(x+2a)}{(x+a)(x+3a)(x+2a)}+\frac{x+3a}{(x+a)(x+2a)(x+3a)}$$

$$=\frac{(x+a)-2(x+2a)+(x+3a)}{(x+a)(x+2a)(x+3a)}=\frac{0}{(x+a)(x+2a)(x+3a)}=0.$$

56. $$\frac{(x-1)^2-2(x-2)^2+(x-3)^2}{(x-1)(x-2)(x-3)}$$
$$=\frac{(x^2-2x+1)-2(x^2-4x+4)+(x^2-6x+9)}{(x-1)(x-2)(x-3)}=\frac{2}{(x-1)(x-2)(x-3)}.$$

57. We have
$$\frac{1}{x(x-1)}-\frac{2}{(x-1)(x+1)}+\frac{1}{x(x+1)}=\frac{(x+1)-2x+(x-1)}{x(x-1)(x+1)}=0.$$

58. $$\frac{2a}{(x-2a)^2}-\frac{(x-a)}{(x-2a)(x-3a)}+\frac{2}{x-3a}$$
$$=\frac{2a(x-3a)-(x-a)(x-2a)+2(x-2a)^2}{(x-2a)^2(x-3a)}$$
$$=\frac{(2ax-6a^2)-(x^2-3ax+2a^2)+(2x^2-8ax+8a^2)}{(x-2a)^2(x-3a)}=\frac{x^2-3ax}{(x-2a)^2(x-3a)}=\frac{x}{(x-2a)^2}.$$

59. $$\frac{3+2a}{2-a}-\frac{2-3a}{2+a}+\frac{a^2-16a}{4-a^2}=\frac{(3+2a)(2+a)-(2-3a)(2-a)+(a^2-16a)}{(2-a)(2+a)}$$
$$=\frac{(6+7a+2a^2)-(4-8a+3a^2)+(a^2-16a)}{(2-a)(2+a)}=\frac{2-a}{(2-a)(2+a)}=\frac{1}{2+a}.$$

60. $$\frac{(x+ay)^2-(x-ay)^2+2(x^2+a^2y^2)}{(x-ay)(x+ay)}$$
$$=\frac{(x^2+2axy+a^2y^2)-(x^2-2axy+a^2y^2)+(2x^2+2a^2y^2)}{(x-ay)(x+ay)}$$
$$=\frac{2(x+ay)^2}{(x-ay)(x+ay)}=\frac{2(x+ay)}{(x-ay)}.$$

61. $$\frac{x^2-2x}{x^2-1}-\frac{x+3}{x+1}+\frac{4x}{x-1}=\frac{(x^2-2x)-(x+3)(x-1)+4x(x+1)}{(x-1)(x+1)}$$
$$=\frac{(x^2-2x)-(x^2+2x-3)+(4x^2+4x)}{(x-1)(x+1)}=\frac{4x^2+3}{x^2-1}.$$

62. $$\frac{(1+x)+(1-x)+(1+2x-x^2)}{(1-x)(1+x)}=\frac{3+2x-x^2}{(1-x)(1+x)}=\frac{(3-x)(1+x)}{(1-x)(1+x)}=\frac{3-x}{1-x}.$$

63. $$\frac{1}{x+3y}+\frac{6y}{(x-3y)(x+3y)}+\frac{1}{x-3y}=\frac{(x-3y)+6y+(x+3y)}{(x-3y)(x+3y)}$$
$$=\frac{2x+6y}{(x-3y)(x+3y)}=\frac{2(x+3y)}{(x-3y)(x+3y)}=\frac{2}{x-3y}.$$

64. $$\frac{\{x-(y-z)\}\,\{x+(y-z)\}}{\{(x+y)-z\}\,\{(x+y)+z\}}+\frac{\{y-(z-x)\}\,\{y+(z-x)\}}{\{(y+z)-x\}\,\{(y+z)+x\}}$$
$$+\frac{\{z-(x-y)\}\,\{z+(x-y)\}}{\{(z+x)-y\}\,\{(z+x)+y\}}=\frac{(x-y+z)\,(x+y-z)}{(x+y-z)\,(x+y+z)}+\frac{(x+y-z)\,(-x+y+z)}{(-x+y+z)\,(x+y+z)}$$
$$+\frac{(-x+y+z)\,(x-y+z)}{(x-y+z)\,(x+y+z)}=\frac{x-y+z}{x+y+z}+\frac{x+y-z}{x+y+z}+\frac{-x+y+z}{x+y+z}$$
$$=\frac{(x-y+z)+(x+y-z)+(-x+y+z)}{x+y+z}=\frac{x+y+z}{x+y+z}=1.$$

65. $$\frac{\{a-(b+c)\}\,\{a+(b+c)\}}{\{(a+b)-c\}\,\{(a+b)+c\}}+\frac{\{b-(c+a)\}\,\{b+(c+a)\}}{\{(b-c)-a\}\,\{(b-c)+a\}}$$
$$+\frac{\{c-(a-b)\}\,\{c+(a-b)\}}{\{(c-a)-b\}\,\{(c-a)+b\}}=\frac{(a-b-c)\,(a+b+c)}{(a+b-c)\,(a+b+c)}+\frac{(-a+b-c)\,(a+b+c)}{(-a+b-c)\,(a+b-c)}$$
$$+\frac{(-a+b+c)\,(a-b+c)}{(-a-b+c)\,(-a+b+c)}=\frac{a-b-c}{a+b-c}+\frac{a+b+c}{a+b-c}+\frac{a-b+c}{-a-b+c}$$
$$=\frac{a-b-c}{a+b-c}+\frac{a+b+c}{a+b-c}+\frac{-a+b-c}{a+b-c}$$
$$=\frac{(a-b-c)+(a+b+c)+(-a+b-c)}{a+b-c}=\frac{a+b-c}{a+b-c}=1.$$

XXXIX. PAGE 137.

13. $\dfrac{2a}{5x}\div\dfrac{3a^2b}{5x^2y}=\dfrac{2a}{5x}\times\dfrac{5x^2y}{3a^2b}=\dfrac{2xy}{3ab}$. 14. $\dfrac{x^2y}{a^2z^2}\div\dfrac{b^2x}{a^2z}=\dfrac{x^2y}{a^2z^2}\times\dfrac{a^2z}{b^2x}=\dfrac{xy}{b^2z}$.

15. $\dfrac{x-y}{x\,(x+y)}\times\dfrac{x+y}{y\,(x-y)}=\dfrac{(x-y)\,(x+y)}{xy\,(x-y)\,(x+y)}=\dfrac{1}{xy}$.

16. $\dfrac{a+b}{a^2\,(a-b)}\times\dfrac{b\,(a-b)}{a\,(a+b)}=\dfrac{b\,(a+b)\,(a-b)}{a^3\,(a+b)\,(a-b)}=\dfrac{b}{a^3}$.

17. $\dfrac{x\,(x+2)}{(x-3)\,(x+3)}\times\dfrac{x\,(x-3)}{(x-2)\,(x+2)}=\dfrac{x^2}{(x+3)\,(x-2)}$.

18. $\dfrac{(x-y)\,(x+y)}{(x-2y)\,(x+2y)}\times\dfrac{x-2y}{x+y}=\dfrac{x-y}{x+2y}$.

19. $\dfrac{x^3+3x^2}{x+4}\times\dfrac{x^2+4x}{x+3}=\dfrac{x^2\,(x+3)}{x+4}\times\dfrac{x\,(x+4)}{x+3}=x^3$.

20. $\dfrac{a+4b}{a^2+5ab}\times\dfrac{a^3+5a^2b}{ab+4b^2}=\dfrac{a+4b}{a\,(a+5b)}\times\dfrac{a^2\,(a+5b)}{b\,(a+4b)}=\dfrac{a}{b}$.

21. $\dfrac{(x-1)\,(x-2)\,(x-3)}{(x-2)\,(x-3)\,(x-4)}=\dfrac{x-1}{x-4}$.

22. $\dfrac{(x+1)\,(x+2)(x+3)}{(x+1)^2\,(x+2)^2\,(x+3)^2}=\dfrac{1}{(x+1)\,(x+2)\,(x+3)}$.

23. $\dfrac{(x-1)\,(x-2)}{(x-2)\,(x-3)}\times\dfrac{(x-3)\,(x-4)}{(x-1)\,(x-4)}=\dfrac{(x-1)\,(x-2)\,(x-3)\,(x-4)}{(x-1)\,(x-2)\,(x-3)\,(x-4)}=1$.

24. $\dfrac{(x-1)(x+1)}{(x+5)(x-2)} \times \dfrac{(x-5)(x+5)}{(x-4)(x+1)} = \dfrac{(x-1)(x-5)}{(x-2)(x-4)}.$

25. $\dfrac{(x-a)(x^2+ax+a^2)}{(x-2a)(x+2a)} \times \dfrac{x+2a}{x-a} = \dfrac{x^2+ax+a^2}{x-2a}.$

26. $\dfrac{a^3-x^3}{a^3+x^3} \times \dfrac{a^2-x^2}{(a-x)^2} = \dfrac{(a-x)(a^2+ax+x^2)}{(a+x)(a^2-ax+x^2)} \times \dfrac{(a-x)(a+x)}{(a-x)^2} = \dfrac{a^2+ax+x^2}{a^2-ax+x^2}.$

27. $\dfrac{a+x}{(a-x)^2} \times \dfrac{(a-x)(a+x)}{a^2+x^2} \times \dfrac{(a-x)(a+x)(a^2+x^2)}{(a+x)^3}$

$$= \frac{(a-x)^2(a+x)^3(a^2+x^2)}{(a-x)^2(a+x)^3(a^2+x^2)} = 1.$$

28. $\dfrac{(a+b)^2}{(a-b)^3} \times \dfrac{\{(a-b)(a+b)\}^2}{(a-b)(a+b)(a^2+b^2)} \times \dfrac{a^2+b^2}{(a+b)^3} = \dfrac{(a-b)^2(a+b)^4(a^2+b^2)}{(a-b)^4(a+b)^4(a^2+b^2)}$

$$= \frac{1}{(a-b)^2}.$$

29. $\dfrac{\{(a-b)-c\}\{(a-b)+c\}}{\{(a-c)-b\}\{(a-c)+b\}} \times \dfrac{\{b-(c-a)\}\{b+(c-a)\}}{\{c-(a-b)\}\{c+(a-b)\}}$

$$= \frac{(a-b-c)(a-b+c)}{(a-b-c)(a+b-c)} \times \frac{(a+b-c)(-a+b+c)}{(-a+b+c)(a-b+c)} = 1.$$

30. $\dfrac{x^2-(y+z)^2}{x^2-(y-z)^2} \times \dfrac{y^2-(x-z)^2}{y^2-(x+z)^2} = \dfrac{\{x-(y+z)\}\{x+(y+z)\}}{\{x-(y-z)\}\{x+(y-z)\}}$

$$\times \frac{\{y-(x-z)\}\{y+(x-z)\}}{\{y-(x+z)\}\{y+(x+z)\}} = \frac{(x-y-z)(x+y+z)}{(x-y+z)(x+y-z)} \times \frac{(-x+y+z)(x+y-z)}{(-x+y-z)(x+y+z)}$$

$$= \frac{(x-y-z)(-x+y+z)}{(-x+y-z)(x-y+z)} = \frac{(-x+y+z)^2}{(x-y+z)^2}.$$

31. $\dfrac{(x^2+y^2)(x^4-x^2y^2+y^4)}{(x-y)(x+y)(x^4+x^2y^2+y^4)} \times \dfrac{x-y}{x+y} \times \dfrac{x^4+x^2y^2+y^4}{x^4-x^2y^2+y^4} = \dfrac{x^2+y^2}{(x+y)^2}.$

32. $\left(\dfrac{x^2+y^2}{x^2+y^2} - \dfrac{2xy}{x^2+y^2}\right) \div \left(\dfrac{x^3-y^3}{x-y} - \dfrac{3xy(x-y)}{x-y}\right)$

$$= \frac{x^2+y^2-2xy}{x^2+y^2} \div \frac{x^3-y^3-3xy(x-y)}{x-y} = \frac{(x-y)^2}{x^2+y^2} \times \frac{x-y}{x^3-3x^2y+3xy^2-y^3}$$

$$= \frac{(x-y)^2}{x^2+y^2} \times \frac{(x-y)}{(x-y)^3} = \frac{1}{x^2+y^2}.$$

XL. PAGES 139, 140.

1. $\dfrac{x+2a}{a-2x} - \dfrac{a+2x}{x-2a} = \dfrac{(2a+x)(2a-x)+(a+2x)(a-2x)}{(a-2x)(2a-x)} = \dfrac{5(a^2-x^2)}{(a-2x)(2a-x)}.$

$$\frac{3}{2a-x} - \frac{1}{a-x} = \frac{3(a-x)-(2a-x)}{(2a-x)(a-x)} = \frac{a-2x}{(2a-x)(a-x)}.$$

Hence we have $\dfrac{5(a-x)(a+x)}{(a-2x)(2a-x)} \times \dfrac{(a-2x)}{(2a-x)(a-x)} = \dfrac{5(a+x)}{(2a-x)^2}.$

2. $$\frac{x}{1+x}+\frac{1-x}{x}=\frac{x^2+(1-x)(1+x)}{x(1+x)}=\frac{1}{x(1+x)}.$$
$$\frac{x}{1+x}-\frac{1-x}{x}=\frac{x^2-(1-x)(1+x)}{x(1+x)}=\frac{2x^2-1}{x(1+x)}.$$

Hence we have $\frac{1}{x(1+x)}\div\frac{2x^2-1}{x(1+x)}=\frac{1}{x(1+x)}\times\frac{x(1+x)}{2x^2-1}=\frac{1}{2x^2-1}$.

3. $$\left(\frac{x^3}{xy}-\frac{y^3}{xy}\right)\left(\frac{x^3}{x^2y^2}+\frac{y^3}{x^2y^2}\right)\div\left(\frac{x^3}{xy}+\frac{y^3}{xy}\right)\left(\frac{x^3}{x^2y^2}-\frac{y^3}{x^2y^2}\right)$$
$$=\frac{x^3-y^3}{xy}\cdot\frac{x^3+y^3}{x^2y^2}\div\left\{\frac{x^3+y^3}{xy}\times\frac{x^3-y^3}{x^2y^2}\right\}=\frac{x^6-y^6}{x^3y^3}\div\frac{x^6-y^6}{x^3y^3}=1.$$

4. $$\frac{2xy}{y-x}+\frac{xy}{x-y}-\frac{xy}{y-x}=\frac{2xy}{y-x}-\frac{xy}{y-x}-\frac{xy}{y-x}=0.$$

5. $$\frac{xy}{y+x}+\frac{xy}{x+y}-\frac{2xy}{y+x}=0.$$

6. $$\frac{x^2+y^2+2xy}{(x+y)xy}+\frac{x^2+y^2-2xy}{(x-y)xy}=\frac{(x+y)^2}{(x+y)xy}+\frac{(x-y)^2}{(x-y)xy}$$
$$=\frac{x+y}{xy}+\frac{x-y}{xy}=\frac{2x}{xy}=\frac{2}{y}.$$

7. $$\frac{a+b}{a+2b}+\frac{b}{a}=\frac{(a+b)a+b(a+2b)}{a(a+2b)}=\frac{a^2+2ab+2b^2}{a(a+2b)}.$$
$$\frac{a+b}{a}-\frac{b}{a+2b}=\frac{(a+b)(a+2b)-ab}{a(a+2b)}=\frac{a^2+2ab+2b^2}{a(a+2b)}.$$

Thus the given fraction is $\frac{a^2+2ab+2b^2}{a(a+2b)}\div\frac{a^2+2ab+2b^2}{a(a+2b)}=1.$

8. $$\frac{x(x-1)+\dfrac{x-1}{x+1}}{x+\dfrac{1}{x+1}}=\frac{\left\{x(x-1)+\dfrac{x-1}{x+1}\right\}(x+1)}{\left\{x+\dfrac{1}{x+1}\right\}(x+1)}=\frac{x(x-1)(x+1)+(x-1)}{x(x+1)+1}$$
$$=\frac{(x-1)\{x(x+1)+1\}}{x(x+1)+1}=x-1.$$

9. $$\frac{x+2}{2x+3}-\frac{4x+5}{5x+6}=\frac{(x+2)(5x+6)-(4x+5)(2x+3)}{(2x+3)(5x+6)}=\frac{-3x^2-6x-3}{(2x+3)(5x+6)}$$
$$=\frac{-3(x+1)^2}{(2x+3)(5x+6)}.$$
$$\frac{2x+3}{3x+4}-\frac{3x+4}{4x+5}=\frac{(2x+3)(4x+5)-(3x+4)(3x+4)}{(3x+4)(4x+5)}=\frac{-(x+1)^2}{(3x+4)(4x+5)}.$$

Whence given fraction is equal to
$$\frac{-3(x+1)^2}{(2x+3)(5x+6)}\div\frac{-(x+1)^2}{(3x+4)(4x+5)}=\frac{3(3x+4)(4x+5)}{(2x+3)(5x+6)}.$$

10. $$\frac{1+x}{1+x^2}-\frac{1+x^2}{1+x^3}=\frac{(1+x)(1+x^3)-(1+x^2)^2}{(1+x^2)(1+x^3)}$$
$$=\frac{x-2x^2+x^3}{(1+x^2)(1+x^3)}=\frac{x(1-x)^2}{(1+x^2)(1+x^3)}.$$

Also $$\frac{1+x^2}{1+x^3}-\frac{1+x^3}{1+x^4}=\frac{(1+x^2)(1+x^4)-(1+x^3)^2}{(1+x^3)(1+x^4)}$$
$$=\frac{x^2-2x^3+x^4}{(1+x^3)(1+x^4)}=\frac{x^2(1-x)^2}{(1+x^3)(1+x^4)}.$$

Whence given fraction is equal to
$$\frac{x(1-x)^2}{(1+x^2)(1+x^3)}\div\frac{x^2(1-x)^2}{(1+x^3)(1+x^4)}=\frac{1+x^4}{x(1+x^2)}.$$

11. $$\frac{1}{x}-\frac{1}{x+\cfrac{x}{\left(3x-\frac{1}{x}\right)x}}=\frac{1}{x}-\frac{1}{x+\cfrac{x}{3x^2-1}}=\frac{1}{x}-\frac{3x^2-1}{x(3x^2-1)+x}$$
$$=\frac{1}{x}-\frac{3x^2-1}{3x^3}=\frac{3x^2}{3x^3}-\frac{3x^2-1}{3x^3}=\frac{1}{3x^3}.$$

12. $$x^3-\frac{x^2}{1+\cfrac{(1-x)}{\left(x+\frac{1}{x}\right)}}=x^3-\frac{x^2}{1+\cfrac{(1-x)x}{x^2+1}}=x^3-\frac{x^2(x^2+1)}{x^2+1+(1-x)x}$$
$$=x^3-\frac{x^2(x^2+1)}{x+1}=\frac{x^3(x+1)-x^2(x^2+1)}{x+1}=\frac{x^3-x^2}{x+1}.$$

13. Fraction $$=\frac{1}{x+\cfrac{3-a}{(3-a)+(a+1)}}=\frac{1}{x+\cfrac{3-a}{4}}=\frac{4}{4x+3-a}.$$

14. Fraction $$=\frac{1}{x^2-\cfrac{(x^3+1)(x-1)}{x(x-1)+1}}=\frac{1}{x^2-\cfrac{(x^2-x+1)(x+1)(x-1)}{x^2-x+1}}$$
$$=\frac{1}{x^2-(x+1)(x-1)}=1.$$

XLI. PAGES 142—145.

4. $$\frac{(x-5y)(x-y)}{x(x-4y)(x-y)}=\frac{x-5y}{x(x-4y)}.$$

5. $$\frac{x(x-11y)(x+y)}{y(x-9y)(x+y)}=\frac{x(x-11y)}{y(x-9y)}.$$

6. $$\frac{\{(a-b)(a+b)\}\{(a-b)(a^2+ab+b^2)\}}{(a-b)\{(a-b)(a+b)(a^2+b^2)\}}=\frac{(a^2+ab+b^2)}{(a^2+b^2)}.$$

7. $$\frac{(a-b)(a+b)(a^2-b^2)(a^4+a^2b^2+b^4)}{(a+b)^2(a-b)^2}=a^4+a^2b^2+b^4.$$

8. The H. C. F. will be found to be x^2+4x+3.

Then $$\frac{x^3+3x^2-x-3}{x^4+4x^3-12x-9}=\frac{(x^2+4x+3)(x-1)}{(x^2+4x+3)(x^2-3)}=\&c.$$

9. Find H. C. F. of numerator and denominator.

$$2x^4+3x^3-2x^2+12x+5\ )\ 3x^4+5x^3-7x^2+2x+2$$
$$2$$
$$6x^4+10x^3-14x^2+4x+4\ (\ 3$$
$$6x^4+9x^3-6x^2+36x+15$$
$$x^3-8x^2-32x-11\ )\ 2x^4+3x^3-2x^2+12x+5\ (\ 2x+19$$
$$2x^4-16x^3-64x^2-22x$$
$$19x^3+62x^2+34x+5$$
$$19x^3-152x^2-608x-209$$
$$214\ )\ 214x^2+642x+214$$
$$x^2+3x+1\ )\ x^3-8x^2-32x-11\ (\ x-11$$
$$x^3+3x^2+x$$
$$-11x^2-33x-11$$
$$-11x^2-33x-11$$

Then $\dfrac{3x^4+5x^3-7x^2+2x+2}{2x^4+3x^3-2x^2+12x+5}=\dfrac{(x^2+3x+1)(3x^2-4x+2)}{(x^2+3x+1)(2x^2-3x+5)}=$ &c.

10.

$$3x^4+14x^3y+9xy^3+2y^4$$
$$2$$
$$2x^4+9x^3y+14xy^3+3y^4\ )\ 6x^4+28x^3y+18xy^3+4y^4\ (\ 3$$
$$6x^4+27x^3y+42xy^3+9y^4$$
$$y\ )\ x^3y-24xy^3-5y^4$$
$$x^3-24xy^2-5y^3\ )\ 2x^4+9x^3y+14xy^3+3y^4\ (\ 2x+9y$$
$$2x^4-48x^2y^2-10xy^3$$
$$9x^3y+48x^2y^2+24xy^3+3y^4$$
$$9x^3y-216xy^3-45y^4$$
$$48y^2\ )\ 48x^2y^2+240xy^3+48y^4$$
$$x^2+5xy+y^2\ )\ x^3-24xy^2-5y^3\ (\ x-5y$$
$$x^3+5x^2y+xy^2$$
$$-5x^2y-25xy^2-5y^3$$
$$-5x^2y-25xy^2-5y^3$$

Then $\dfrac{2x^4+9x^3y+14xy^3+3y^4}{3x^4+14x^3y+9xy^3+2y^4}=\dfrac{(x^2+5xy+y^2)(2x^2-xy+3y^2)}{(x^2+5xy+y^2)(3x^2-xy+2y^2)}=$ &c.

11. $$\begin{array}{rl} 2x^3-11x^2y+11xy^2+4y^3\;) & 2x^4-3x^3y+7x^2y^2-12xy^3-4y^4\;(\;x+4y \\ & \underline{2x^4-11x^3y+11x^2y^2+4xy^3} \\ & \quad 8x^3y-4x^2y^2-16xy^3-4y^4 \\ & \quad \underline{8x^3y-44x^2y^2+44xy^3+16y^4} \\ & \quad 20y^2\;)\;\underline{40x^2y^2-60xy^3-20y^4} \\ & \qquad\qquad 2x^2-3xy-y^2 \end{array}$$

$$\begin{array}{rl} 2x^2-3xy-y^2\;) & 2x^3-11x^2y+11xy^2+4y^3\;(\;x-4y \\ & \underline{2x^3-3x^2y-xy^2} \\ & -8x^2y+12xy^2+4y^3 \\ & -8x^2y+12xy^2+4y^3 \end{array}$$

$$\frac{2x^3-11x^2y+11xy^2+4y^3}{2x^4-3x^3y+7x^2y^2-12xy^3-4y^4}=\frac{(2x^2-3xy-y^2)(x-4y)}{(2x^2-3xy-y^2)(x^2+4y^2)}=\&c.$$

12. $\frac{2(x+y)}{3}-\frac{3(x-y)}{5}=\frac{10(x+y)}{15}-\frac{9(x-y)}{15}=\frac{x+19y}{15}.$

13. $\frac{5(x-2y)}{4}-\frac{4(y-2x)}{5}=\frac{25(x-2y)}{20}-\frac{16(y-2x)}{20}=\frac{57x-66y}{20}.$

14. $\frac{1}{4-x}-\frac{4}{16-x^2}=\frac{4+x-4}{16-x^2}=\frac{x}{16-x^2}.$

15. $\frac{1}{3-x}-\frac{3}{9-x^2}=\frac{3+x}{(3-x)(3+x)}-\frac{3}{9-x^2}=\frac{3+x-3}{9-x^2}=\frac{x}{9-x^2}.$

16. $\frac{x}{x-2y}-\frac{x^2}{(x-2y)^2}=\frac{x(x-2y)-x^2}{(x-2y)^2}=\frac{-2xy}{(x-2y)^2}.$

17. $\frac{a}{a-4b}+\frac{4ab}{(a-4b)^2}=\frac{a(a-4b)+4ab}{(a-4b)^2}=\frac{a^2}{(a-4b)^2}.$

18. $$\left(\frac{1}{a+4}-\frac{1}{a+5}\right)+\left(\frac{1}{a+6}-\frac{1}{a+7}\right)=\frac{(a+5)-(a+4)}{(a+4)(a+5)}+\frac{(a+7)-(a+6)}{(a+6)(a+7)}$$

$$=\frac{1}{(a+4)(a+5)}+\frac{1}{(a+6)(a+7)}=\frac{(a+6)(a+7)+(a+4)(a+5)}{(a+4)(a+5)(a+6)(a+7)}=\&c.$$

19. $$\left(\frac{1}{a}-\frac{1}{a+9}\right)-\left(\frac{3}{a+3}-\frac{3}{a+6}\right)=\frac{(a+9)-a}{a(a+9)}-\frac{3(a+6)-3(a+3)}{(a+3)(a+6)}$$

$$=\frac{9}{a(a+9)}-\frac{9}{(a+3)(a+6)}=\frac{9(a+3)(a+6)-9a(a+9)}{a(a+3)(a+6)(a+9)}$$

$$=\frac{9(a^2+9a+18)-9(a^2+9a)}{a(a+3)(a+6)(a+9)}=\frac{162}{a(a+3)(a+6)(a+9)}.$$

20. $$\frac{(x-y)(x^2+xy+y^2)}{(x-3y)(x+3y)}\times\frac{x+3y}{x-y}=\frac{x^2+xy+y^2}{x-3y}.$$

21. $$\frac{x^2-y^2}{x^2-4y^2}\times\frac{2y-x}{y-x}=\frac{(x-y)(x+y)}{(x-2y)(x+2y)}\times\frac{x-2y}{x-y}=\frac{x+y}{x+2y}.$$

22. $$\left(\frac{1}{a}+\frac{1}{b}\right)^2-\frac{1}{ab}-\frac{4}{(a-b)^2}=\frac{(a+b)^2}{a^2b^2}-\frac{1}{ab}-\frac{4}{(a-b)^2}$$

$$=\frac{(a+b)^2(a-b)^2-ab(a-b)^2-4a^2b^2}{a^2b^2(a-b)^2}$$

$$=\frac{a^4-2a^2b^2+b^4-a^3b+2a^2b^2-ab^3-4a^2b^2}{a^2b^2(a-b)^2}=\frac{a^4-a^3b-4a^2b^2-ab^3+b^4}{a^2b^2(a-b)^2}.$$

Also $$\left(\frac{1}{a}-\frac{1}{b}\right)^2-\frac{1}{ab}=\frac{(a-b)^2-ab}{a^2b^2}=\frac{a^2-3ab+b^2}{a^2b^2}.$$

Hence we have $$\frac{a^4-a^3b-4a^2b^2-ab^3+b^4}{a^2b^2(a-b)^2}\times\frac{a^2b^2}{a^2-3ab+b^2}=\frac{a^2+2ab+b^2}{(a-b)^2},$$

since $$(a^4-a^3b-4a^2b^2-ab^3+b^4)\div(a^2-3ab+b^2)=a^2+2ab+b^2.$$

23. $$\frac{x(x-8)+5(x-2)}{(x-2)(x-8)}\times\frac{(x-3)(x+2)-2(3x-8)}{(3x-8)(x+2)}=\frac{x^2-3x-10}{(x-2)(x-8)}$$

$$\times\frac{x^2-7x+10}{(3x-8)(x+2)}=\frac{(x-5)(x+2)}{(x-2)(x-8)}\times\frac{(x-5)(x-2)}{(3x-8)(x+2)}=\&\text{c}.$$

24. $$\frac{(x^4-4x^3y+6x^2y^2-4xy^3+y^4)-(x^3y-2x^2y^2+xy^3)-2x^2y^2}{(x^4-x^3y-xy^3+y^4)+2x^2y^2}$$

$$=\frac{x^4-5x^3y+6x^2y^2-5xy^3+y^4}{x^4-x^3y+2x^2y^2-xy^3+y^4}.$$

The H. C. F. will be found to be x^2-xy+y^2, and the fraction

$$=\frac{(x^2-xy+y^2)(x^2-4xy+y^2)}{(x^2-xy+y^2)(x^2+y^2)}=\&\text{c}.$$

25. $$\frac{x(x^2+x+1)+x(x-1)(2x+1)-3(x-1)(x^2+x+1)}{x(x-1)(x^2+x+1)}=\frac{3}{x(x^3-1)}.$$

26. $$\frac{1}{2(3a-b)}+\frac{1}{3a+2b}-\frac{3}{2(3a+b)}$$
$$=\frac{(3a+2b)(3a+b)+2(3a-b)(3a+b)-3(3a+2b)(3a-b)}{2(3a-b)(3a+b)(3a+2b)}$$
$$=\frac{(9a^2+9ab+2b^2)+(18a^2-2b^2)-(27a^2+9ab-6b^2)}{2(3a-b)(3a+b)(3a+2b)}$$
$$=\frac{6b^2}{2(3a-b)(3a+b)(3a+2b)}.$$

27. $$\frac{1}{2(4a+b)}+\frac{1}{2(4a-b)}+\frac{b}{(4a-b)(4a+b)}$$
$$=\frac{(4a-b)+(4a+b)+2b}{2(4a-b)(4a+b)}=\frac{8a+2b}{(4a-b)(8a+2b)}=\&c.$$

28. $$\frac{(a^2+b^2)(a^4-a^2b^2+b^4)}{(a^2-b^2)(a^4+a^2b^2+b^4)}\times\frac{a-b}{a^2+b^2}\times\frac{a^4+a^2b^2+b^4}{a^4-a^2b^2+b^4}=\frac{a-b}{a^2-b^2}=\frac{1}{a+b}.$$

29. $$\frac{\left(\frac{m^2+n^2}{n}-m\right)mn}{\left(\frac{1}{n}-\frac{1}{m}\right)mn}\times\frac{m^2-n^2}{m^3+n^3}=\frac{(m^2+n^2-mn)m}{m-n}\times\frac{m^2-n^2}{m^3+n^3}$$
$$=\frac{m(m^2-mn+n^2)}{m-n}\times\frac{(m-n)(m+n)}{(m+n)(m^2-mn+n^3)}=m.$$

30. $$\frac{\left\{1+\frac{c}{a+b}+\left(\frac{c}{a+b}\right)^2\right\}\left\{\left(1-\frac{c}{a+b}\right)\left(1+\frac{c}{a+b}\right)\right\}}{\left(1-\frac{c}{a+b}\right)\left[1+\frac{c}{a+b}+\left(\frac{c}{a+b}\right)^2\right]\left\{1+\frac{c}{a+b}\right\}}=1.$$

31. $$\frac{(x+y)^2-(x-y)^2-4y^2}{(x-y)(x+y)}=\frac{4xy-4y^2}{(x-y)(x+y)}=\frac{4y(x-y)}{(x-y)(x+y)}=\&c.$$

32. $$\left\{\frac{a(a-3b)+2b^2}{a(a-3b)}\right\}\left\{\frac{(2b-a)+b}{2b-a}\right\}=\frac{a^2-3ab+2b^2}{a(a-3b)}\cdot\frac{3b-a}{2b-a}$$
$$=\frac{(a-b)(a-2b)}{a(a-3b)}\cdot\frac{a-3b}{a-2b}=\&c.$$

33. $$\frac{-(x^3+y^3)}{xy(x-y)(x+y)}-\frac{y}{x(x+y)}+\frac{x}{y(x-y)}=\frac{-(x^3+y^3)-y^2(x-y)+x^2(x+y)}{xy(x-y)(x+y)}$$
$$=\frac{x^2y-xy^2}{xy(x-y)(x+y)}=\frac{xy(x-y)}{xy(x-y)(x+y)}=\frac{1}{x+y}.$$

34. $$\frac{a+b}{a+5b}+\frac{b^2-a^2}{a^2-25b^2}+\frac{a-b}{a-5b}=\frac{(a+b)(a-5b)+(b^2-a^2)+(a-b)(a+5b)}{(a-5b)(a+5b)}$$
$$=\frac{(a^2-4ab-5b^2)+(b^2-a^2)+(a^2+4ab-5b^2)}{a^2-25b^2}=\&c.$$

35. $$\frac{a-b}{a-3b}+\frac{a+b}{a+3b}-\frac{a^2-b^2}{a^2-9b^2}=\frac{(a-b)(a+3b)+(a+b)(a-3b)-(a^2-b^2)}{(a-3b)(a+3b)}$$
$$=\frac{(a^2+2ab-3b^2)+(a^2-2ab-3b^2)-(a^2-b^2)}{a^2-9b^2}=\text{\&c.}$$

36. $$\frac{1}{2(x-1)}-\frac{x-5}{(x-5)(x-2)}+\frac{1}{2}\,\frac{x-6}{(x-6)(x-3)}$$
$$=\frac{1}{2(x-1)}-\frac{1}{x-2}+\frac{1}{2(x-3)}$$
$$=\frac{(x-2)(x-3)-2(x-3)(x-1)+(x-1)(x-2)}{2(x-1)(x-2)(x-3)}=\frac{1}{(x-1)(x-2)(x-3)}.$$

37. We have
$$\frac{3(x-4)-2(x-3)}{(x-3)(x-4)}-\frac{x-6}{(x-2)(x-5)}=\frac{x-6}{(x-3)(x-4)}-\frac{x-6}{(x-2)(x-5)}$$
$$=\frac{(x-6)\{(x-2)(x-5)-(x-3)(x-4)\}}{(x-2)(x-3)(x-4)(x-5)}=\frac{-2(x-6)}{(x-2)(x-3)(x-4)(x-5)}.$$

38. $$\frac{3(x-5)-2(x-4)}{(x-4)(x-5)}-\frac{x-7}{(x-6)(x-3)}=\frac{x-7}{(x-4)(x-5)}-\frac{x-7}{(x-6)(x-3)}$$
$$=\frac{(x-7)\{(x-6)(x-3)-(x-4)(x-5)\}}{(x-3)(x-4)(x-5)(x-6)}=\frac{-2(x-7)}{(x-3)(x-4)(x-5)(x-6)}.$$

39. $$\left\{x-y-\frac{x-y}{(x-y)^2+xy}\right\}\times\frac{x^3+y^3}{x^2-y^2}=\frac{(x-y)(x^2-xy+y^2-1)}{x^2-xy+y^2}$$
$$\times\frac{(x+y)(x^2-xy+y^2)}{(x-y)(x+y)}=x^2-xy+y^2-1.$$

40. $$\left\{x+y-\frac{x+y}{(x+y)^2-xy}\right\}\times\frac{x^3-y^3}{x^2-y^2}=\frac{(x+y)(x^2+xy+y^2-1)}{x^2+xy+y^2}$$
$$\times\frac{(x-y)(x^2+xy+y^2)}{(x+y)(x-y)}=x^2+xy+y^2-1.$$

41. $$\frac{1-x^2-(1-x)^2+1+2x^2}{1-x^2}\times\frac{x+1}{2x+1}=\frac{1+2x}{1-x^2}\times\frac{1+x}{1+2x}=\frac{1}{1-x}.$$

42. $$\frac{\left(x-2-\dfrac{1}{x-2}\right)(x-2)(x-5)}{\left(x-2-\dfrac{4}{x-5}\right)(x-2)(x-5)}\times\frac{\left(x-4-\dfrac{4}{x-4}\right)(x-4)}{\left(x-4-\dfrac{1}{x-4}\right)(x-4)}$$
$$=\frac{\{(x-2)^2-1\}(x-5)}{\{(x-2)(x-5)-4\}(x-2)}\times\frac{(x-4)^2-4}{(x-4)^2-1}=\frac{\{(x-2)-1\}\{(x-2)+1\}(x-5)}{\{x^2-7x+6\}(x-2)}$$
$$\times\frac{\{(x-4)-2\}\{(x-4)+2\}}{\{(x-4)-1\}\{(x-4)+1\}}=\frac{(x-3)(x-1)(x-5)}{(x-1)(x-6)(x-2)}\times\frac{(x-6)(x-2)}{(x-5)(x-3)}=1.$$

43. $$\frac{6}{(x-2)(x+4)} - \frac{7}{(x-3)(x+4)} + \frac{1}{(x-2)(x-3)}$$

$$= \frac{6(x-3) - 7(x-2) + (x+4)}{(x-2)(x-3)(x+4)} = \frac{(6x-18) - (7x-14) + (x+4)}{(x-2)(x-3)(x+4)} = 0.$$

44. $$\frac{x^2+y^2}{x^2-y^2} - \frac{x^2-y^2}{x^2+y^2} = \frac{(x^2+y^2)^2 - (x^2-y^2)^2}{(x^2-y^2)(x^2+y^2)} = \frac{4x^2y^2}{(x^2-y^2)(x^2+y^2)}$$

$$\frac{x+y}{x-y} - \frac{x-y}{x+y} = \frac{(x+y)^2 - (x-y)^2}{(x-y)(x+y)} = \frac{4xy}{(x-y)(x+y)}.$$

Thus we have

$$(x^3+y^3)\left\{\frac{4x^2y^2}{(x^2-y^2)(x^2+y^2)}\right\} \times \frac{1}{x^2-xy+y^2} \div \left\{\frac{4xy}{(x-y)(x+y)}\right\}$$

$$= (x+y)(x^2-xy+y^2) \times \frac{4x^2y^2}{(x-y)(x+y)(x^2+y^2)} \times \frac{1}{x^2-xy+y^2} \times \frac{(x-y)(x+y)}{4xy}$$

$$= \frac{xy(x+y)}{x^2+y^2}.$$

45. $$\frac{(a^2+b^2) - 2ab}{2ab} \times \frac{ab^2}{a^3+b^3} \times \frac{a^2-ab+b^2}{4ab(a+b)}$$

$$= \frac{(a-b)^2}{2ab} \times \frac{ab^2}{(a+b)(a^2-ab+b^2)} \times \frac{a^2-ab+b^2}{4ab(a+b)} = \frac{(a-b)^2}{8a(a+b)^2}.$$

46. We have $$\left(\frac{a^2}{b^2} + 2 + \frac{b^2}{a^2}\right) + \left(\frac{b^2}{c^2} + 2 + \frac{c^2}{b^2}\right) + \left(\frac{c^2}{a^2} + 2 + \frac{a^2}{c^2}\right)$$

$$- \left(2 + \frac{a^2}{b^2} + \frac{b^2}{a^2} + \frac{a^2}{c^2} + \frac{c^2}{a^2} + \frac{b^2}{c^2} + \frac{c^2}{b^2}\right) = 4.$$

47. $$\left\{\frac{1}{3x^2 - 14xy + 15y^2} + \frac{2}{3x^2 - 2xy - 5y^2}\right\}$$

$$= \frac{1}{(x-3y)(3x-5y)} + \frac{2}{(3x-5y)(x+y)} = \frac{(x+y) + 2(x-3y)}{(x-3y)(3x-5y)(x+y)}$$

$$= \frac{3x-5y}{(x-3y)(3x-5y)(x+y)} = \frac{1}{(x-3y)(x+y)}.$$

$$\left\{\frac{x+y}{x-3y} - \frac{x-y}{x+3y}\right\} = \frac{(x+y)(x+3y) - (x-y)(x-3y)}{(x-3y)(x+3y)} = \frac{8xy}{(x-3y)(x+3y)}.$$

Whence we have $$\left\{\frac{1}{(x-3y)(x+y)}\right\} \div \left\{\frac{8xy}{(x-3y)(x+3y)}\right\}$$

$$= \frac{1}{(x-3y)(x+y)} \times \frac{(x-3y)(x+3y)}{8xy} = \frac{x+3y}{8xy(x+y)}.$$

48.

$$\frac{(b-c)(b+x)(c+x)+(c-a)(a+x)(c+x)+(a-b)(a+x)(b+x)+(b-c)(c-a)(a-b)}{(a+x)\,(b+x)\,(c+x)}$$

$$=\frac{\begin{array}{l}(b^2c-bc^2+b^2x-c^2x+bx^2-cx^2)+(-a^2c+ac^2-a^2x+c^2x-ax^2+cx^2)\\ \quad+(a^2b-ab^2+a^2x-b^2x+ax^2-bx^2)+(-a^2b+a^2c+ab^2-ac^2-b^2c+bc^2)\end{array}}{(a+x)\,(b+x)\,(c+x)}$$

$$=\frac{0}{(a+x)\,(b+x)\,(c+x)}=0.$$

49. We have $\left\{\frac{a-b}{a+b}-1\right\}^2\div\left\{\frac{a+b}{a-b}-1\right\}^2$

$$=\left(\frac{-2b}{a+b}\right)^2\div\left(\frac{2b}{a-b}\right)^2=\frac{(a-b)^2}{(a+b)^2}.$$

50. We have $\left(\frac{a-b}{a+b}+1\right)^3\div\left(\frac{a+b}{a-b}+1\right)^3$

$$=\left(\frac{2a}{a+b}\right)^3\div\left(\frac{2a}{a-b}\right)^3=\frac{(a-b)^3}{(a+b)^3}.$$

51. We have $16\left\{\frac{(x+4)-(x-17)}{(x-17)\,(x+4)}\right\}-21\left\{\frac{(x+7)-(x-9)}{(x+7)\,(x-9)}\right\}$

$$=16\times21\left\{\frac{1}{(x-17)\,(x+4)}-\frac{1}{(x+7)\,(x-9)}\right\}$$

$$=336\,\frac{(x+7)\,(x-9)-(x-17)\,(x+4)}{(x+4)\,(x+7)\,(x-9)\,(x-17)}=\&c.$$

52. $11\,\frac{(x+2)-(x-5)}{(x+2)\,(x-5)}-7\,\frac{(x+4)-(x-7)}{(x+4)\,(x-7)}=\frac{77}{(x+2)\,(x-5)}-\frac{77}{(x+4)\,(x-7)}$

$$=77\,\frac{(x+4)\,(x-7)-(x+2)\,(x-5)}{(x+2)\,(x-5)\,(x+4)\,(x-7)}=\&c.$$

53. $\left(1+\frac{6}{x+1}\right)+\left(1+\frac{6}{x+2}\right)-\left(1+\frac{6}{x+3}\right)-\left(1+\frac{6}{x+4}\right)$

$$=6\left\{\frac{(x+3)-(x+1)}{(x+1)\,(x+3)}+\frac{(x+4)-(x+2)}{(x+2)\,(x+4)}\right\}$$

$$=\frac{12}{(x+1)\,(x+3)}+\frac{12}{(x+2)\,(x+4)}=\frac{12\,(x+2)\,(x+4)+12\,(x+1)\,(x+3)}{(x+1)\,(x+2)\,(x+3)\,(x+4)}.$$

54. $\frac{\{(a+b-c)-d\}\{(a+b-c)+d\}}{\{(a+b)-(c+d)\}\,\{(a+b)+(c+d)\}}+\frac{\{(b+c-a)-d\}\{(b+c-a)+d\}}{\{(b+c)-(a+d)\}\,\{(b+c)+(a+d)\}}$

$$+\frac{\{(c+a-b)-d\}\,\{(c+a-b)+d\}}{\{(c+a)-(b+d)\}\,\{(c+a)+(b+d)\}}=\frac{\{a+b-c-d\}\,\{a+b-c+d\}}{\{a+b-c-d\}\,\{a+b+c+d\}}$$

$$+\frac{\{-a+b+c-d\}\{-a+b+c+d\}}{\{-a+b+c-d\}\{a+b+c+d\}}+\frac{\{a-b+c-d\}\{a-b+c+d\}}{\{a-b+c-d\}\{a+b+c+d\}}$$

$$=\frac{a+b-c+d}{a+b+c+d}+\frac{-a+b+c+d}{a+b+c+d}+\frac{a-b+c+d}{a+b+c+d}$$

$$=\frac{(a+b-c+d)+(-a+b+c+d)+(a-b+c+d)}{a+b+c+d}=\&c.$$

55. $$\frac{(x-a)(y-a)-xy+ax}{axy(x-a)(y-a)}=\frac{(xy-ax-ay+a^2)-xy+ax}{axy(x-a)(y-a)}$$

$$=\frac{-ay+a^2}{axy(x-a)(y-a)}=\frac{-a(y-a)}{axy(x-a)(y-a)}=\frac{-1}{xy(x-a)}=\frac{1}{xy(a-x)}.$$

56. $$\frac{a+x}{x(x-y)(x-z)}+\frac{a+y}{y(y-z)\{-(x-y)\}}+\frac{a+z}{z\{-(x-z)\}\{-(y-z)\}}$$

$$=\frac{a+x}{x(x-y)(x-z)}-\frac{a+y}{y(y-z)(x-y)}+\frac{a+z}{z(x-z)(y-z)}$$

$$=\frac{(a+x)yz(y-z)-(a+y)xz(x-z)+(a+z)xy(x-y)}{xyz(x-y)(x-z)(y-z)}$$

$$=\frac{ayz(y-z)+xyz(y-z)-axz(x-z)-xyz(x-z)+axy(x-y)+xyz(x-y)}{xyz(x-y)(x-z)(y-z)}$$

$$=\frac{a\{yz(y-z)-xz(x-z)+xy(x-y)\}+xyz\{(y-z)-(x-z)+(x-y)\}}{xyz(x-y)(x-z)(y-z)}$$

$$=\frac{a\{y^2z-yz^2-x^2z+xz^2+x^2y-xy^2\}}{xyz(x-y)(x-z)(y-z)}$$

$$=\frac{a\{x^2(y-z)-x(y^2-z^2)+yz(y-z)\}}{xyz(x-y)(x-z)(y-z)}\text{ [Art. 107.]}$$

$$=\frac{a(y-z)\{x^2-x(y+z)+yz\}}{xyz(x-y)(x-z)(y-z)}=\frac{a(y-z)\{(x-y)(x-z)\}}{xyz(x-y)(x-z)(y-z)}=\&c.$$

57. $$\frac{a-2\dfrac{ab}{a+b}}{b-2\dfrac{ab}{a+b}}=\frac{\left(a-2\dfrac{ab}{a+b}\right)(a+b)}{\left(b-2\dfrac{ab}{a+b}\right)(a+b)}=\frac{a(a+b)-2ab}{b(a+b)-2ab}=\frac{a^2-ab}{-ab+b^2}=-\frac{a}{b}.$$

58. $$\frac{\dfrac{4cd}{c+d}+2c}{\dfrac{4cd}{c+d}-2c}+\frac{\dfrac{4cd}{c+d}+2d}{\dfrac{4cd}{c+d}-2d}=\frac{4cd+2c(c+d)}{4cd-2c(c+d)}+\frac{4cd+2d(c+d)}{4cd-2d(c+d)}$$

$$=\frac{2c^2+6cd}{-2c^2+2cd}+\frac{6cd+2d^2}{2cd-2d^2}=-\frac{c+3d}{c-d}+\frac{3c+d}{c-d}=\frac{-(c+3d)+(3c+d)}{c-d}$$

$$=\frac{2c-2d}{c-d}=2.$$

59. Let $\frac{a}{b}=x$; then $\frac{c}{d}=x$, $a=bx$, $c=dx$.

Thus (i) $\frac{a-b}{a-2b}=\frac{bx-b}{bx-2b}=\frac{b(x-1)}{b(x-2)}=\frac{x-1}{x-2}$;

and $\frac{c-d}{c-2d}=\frac{dx-d}{dx-2d}=\frac{d(x-1)}{d(x-2)}=\frac{x-1}{x-2}$.

(ii) $\frac{a^2}{(a+b)^2}=\frac{b^2x^2}{(bx+b)^2}=\frac{b^2x^2}{b^2(x+1)^2}=\frac{x^2}{(x+1)^2}$;

and $\frac{c^2}{(c+d)^2}=\frac{d^2x^2}{(dx+d)^2}=\frac{x^2}{(x+1)^2}$.

(iii) $\frac{a^2-b^2}{a^2+b^2}=\frac{b^2x^2-b^2}{b^2x^2+b^2}=\frac{b^2(x^2-1)}{b^2(x^2+1)}=\frac{x^2-1}{x^2+1}$;

$\frac{c^2-d^2}{c^2+d^2}=\frac{d^2x^2-d^2}{d^2x^2+d^2}=\frac{d^2(x^2-1)}{d^2(x^2+1)}=\frac{x^2-1}{x^2+1}$.

(iv) $\frac{pa^2+qab+rb^2}{la^2+mab+nb^2}=\frac{pb^2x^2+qb^2x+rb^2}{lb^2x^2+mb^2x+nb^2}=\frac{px^2+qx+r}{lx^2+mx+n}$;

$$\frac{pc^2+qcd+rd^2}{lc^2+mcd+nd^2}=\frac{pd^2x^2+qd^2x+rd^2}{ld^2x^2+md^2x+nd^2}=\frac{d^2(px^2+qx+r)}{d^2(lx^2+mx+n)}=\frac{px^2+qx+r}{lx^2+mx+n}.$$

60. Let $\frac{x}{a-b}=t$; then $\frac{y}{b-c}=\frac{z}{c-a}=t$.

$$\therefore x=(a-b)t,\ y=(b-c)t,\ z=(c-a)t;$$

$$\therefore x+y+z=(a-b)t+(b-c)t+(c-a)t=at-bt+bt-ct+ct-at=0.$$

61. By Theorem I. p. 140, each fraction is equal to

$$\frac{(y+z)+(z+x)+(x+y)}{(ay+bz)+(az+bx)+(ax+by)}=\frac{2x+2y+2z}{ax+ay+az+bx+by+bz}$$

$$=\frac{2(x+y+z)}{(a+b)(x+y+z)}=\&c.$$

62. Let each of these fractions be equal to t.

Then $b-c=xt,\ c-a=yt,\ a-b=zt.$

$$\therefore t(x+y+z)=xt+yt+zt=(b-c)+(c-a)+(a-b)=0;$$

thus either $t=0$, or $x+y+z=0$.

Now if $t=0$, xt, yt, and zt must be zero; *i.e.* $b-c$, $c-a$, and $a-b$ must be zero; or $a=b=c$; thus if a, b, and c are unequal, t is not zero; thus we must have $x+y+z=0$.

Again $t(ax+by+cz)=a(b-c)+b(c-a)+c(a-b)=0.$

Hence $ax+by+cz=0$, since we know that t is not zero.

63. Let each fraction be equal to t; then

$$a+b=t(a-b);\ b+c=2t(b-c);\ c+a=3t(c-a).$$

Thus
$$\begin{aligned} 6(a+b)&=6t(a-b)\\ 3(b+c)&=6t(b-c)\\ 2(c+a)&=6t(c-a)\end{aligned}$$

$\therefore$ adding, $6(a+b)+3(b+c)+2(c+a)=0$, or $8a+9b+5c=0$.

64. By Theorem I. p. 140, each fraction is equal to

$$\frac{-a(bz-cy)-b(cx-az)}{-a(b-c)-b(c-a)}=\frac{acy-bcx}{ac-bc}=\frac{ay-bx}{a-b}.$$

65. We have $\dfrac{2y+2z-3x}{a}=\dfrac{2z+2x-3y}{b}=\dfrac{2x+2y-3z}{c}$.

And, by Theorem I. p. 140, each of these fractions is equal to

$$\frac{(2y+2z-3x)+2(2z+2x-3y)+2(2x+2y-3z)}{a+2b+2c}=\frac{5x}{a+2b+2c}.$$

Similarly each fraction can be proved to be equal to

$$\frac{5y}{b+2c+2a},\ \text{or to}\ \frac{5z}{c+2a+2b}.$$

XLII. Pages 148—150.

1. Multiply both sides by 20; then $4(x-2)-5(x-3)=2(x-7)$;
i.e. $4x-8-5x+15=2x-14$, &c.

2. Multiply both sides by 110; then $10(3x-1)-44(x-1)=11(2-x)$;
i.e. $30x-10-44x+44=22-11x$, &c.

3. Multiply by 60; then $10x-12(x-2)=12x-45(x-3)$;
i.e. $10x-12x+24=12x-45x+135$, &c.

4. Multiply by 30; then $10(x-2)-18(x-1)=30x-25(x-2)$;
i.e. $10x-20-18x+18=30x-25x+50$, &c.

5. Multiply by $2(1-3x)$; then $(1-3x)^2+(1+3x)(1-3x)=4$;
i.e. $1-6x+9x^2+1-9x^2=4$, &c. $\therefore 9x^2-9x^2-6x=4-1-1$;

$$\text{or } -6x=2;\ \therefore x=-\frac{1}{3}.$$

6. Multiply by $12(1+x)$; then $2(3-4x)(1+x)-(5-8x)(1+x)=12(1-x)$;
i.e. $6-2x-8x^2-5+3x+8x^2=12-12x$, &c.

7. Multiply by $8(x-3)$; then

$$2(3x-1)(x-3)-16(x+1)=16(x-3)-(3-6x)(x-3);$$

i.e. $6x^2-20x+6-16x-16=16x-48+6x^2-21x+9$, &c.

8. Multiply by $10(x+1)$; then

$$4x(x+1)-30(x-1)=30(x+1)-(1-4x)(x+1);$$

i. e. $4x^2+4x-30x+30=30x+30-1+3x+4x^2$, &c.

9. Multiply by $2(2x+3)(3x+2)$; then $(3x+2)+(2x+3)=4(3x+2)$;
i. e. $3x+2+2x+3=12x+8$; $\therefore$ $3x+2x-12x=8-2-3$, &c.

10. Multiply by $3(x+3)(5x+1)$; then $(5x+1)+6(x+3)=6(5x+1)$;
i. e. $5x+1+6x+18=30x+6$, &c.

11. Multiply by $4(x+3)(4x+1)$; then $(4x+1)-12(x+3)=8(4x+1)$;
i.e. $4x+1-12x-36=32x+8$, &c.

12. Multiply by $2(2x+3)(3x+4)$; then $6(3x+4)+5(3x+4)=7(2x+3)$;
i.e. $18x+24+15x+20=14x+21$, &c.

13. Multiply by $(x+1)(x-2)$; then $4x(x-2)-x(x+1)=3(x+1)(x-2)$;
i.e. $4x^2-8x-x^2-x=3x^2-3x-6$, &c.

14. Multiply by $(x+5)(x+6)$; then $3x(x+5)-x(x+6)=2(x+5)(x+6)$;
i.e. $3x^2+15x-x^2-6x=2x^2+22x+60$, &c.

15. Multiply by $(x-6)(x-7)$; then $6x(x-6)-x(x-7)=5(x-6)(x-7)$;
i.e. $6x^2-36x-x^2+7x=5x^2-65x+210$, &c.

16. Multiply by $(x+3)(x+7)$; then

$$2x(x+7)-4x(x+3)+2(x+3)(x+7)=0;$$

i.e. $2x^2+14x-4x^2-12x+2x^2+20x+42=0$, &c.

17. Multiply by $(x+4)(x+5)(x+6)$; then

$$(x+5)(x+6)+2(x+4)(x+5)=3(x+4)(x+6);$$

i. e. $x^2+11x+30+2x^2+18x+40=3x^2+30x+72$, &c.

18. Multiply by $(x+1)(x+2)(x+3)$; then

$$3(x+2)(x+3)-2(x+1)(x+3)=(x+1)(x+2);$$

i.e. $3x^2+15x+18-2x^2-8x-6=x^2+3x+2$, &c.

19. Multiply by $2(x+2)(x+4)(x+6)$; then

$$5(x+2)(x+6)-3(x+4)(x+6)=2(x+2)(x+4);$$

i.e. $5x^2+40x+60-3x^2-30x-72=2x^2+12x+16$, &c.

20. Multiply by $4x(x+2)(x+3)$; then

$$6x(x+3)-5x(x+2)-(x+2)(x+3)=0;$$

i.e. $6x^2+18x-5x^2-10x-x^2-5x-6=0$, &c.

21. Multiply by $(3x-7)(3x-5)$; then $(2x-5)(3x-5)=(2x-7)(3x-7)$;
i.e. $6x^2-25x+25=6x^2-35x+49$, &c.

22. $(6x-2)(4x+8)=(3x+7)(8x-5)$;
i.e. $24x^2+40x-16=24x^2+41x-35$, &c.

23. $(x-9)(6x-4)=(2x-5)(3x-7)$;
i. e. $6x^2-58x+36=6x^2-29x+35$, &c.

24. $(3-2x)(4-x)=(2x-7)(x-5)$;
$i.e.\ 12-11x+2x^2=2x^2-17x+35$;
$\therefore\ 2x^2-2x^2-11x+17x=35-12$; or $6x=23$, &c.

25. Multiply both sides of the equation by $x(x-1)(x+3)$.
$(x+1)(x+3)x-(x-1)(x-3)x=8(x-1)(x+3)$;
$i.e.\ x^3+4x^2+3x-x^3+4x^2-3x=8x^2+16x-24$, &c.

26. $\dfrac{(x+2)^2-(x-2)^2}{(x-2)(x+2)}=\dfrac{8}{x+1}$; $i.e.\ \dfrac{8x}{(x-2)(x+2)}=\dfrac{8}{x+1}$;
$\therefore\ x(x+1)=(x-2)(x+2)$; $i.e.\ x^2+x=x^2-4$, &c.

27. $(x-1)(x-4)x-(x+2)(x+3)x+10(x+3)(x-4)=0$;
$i.e.\ x^3-5x^2+4x-x^3-5x^2-6x+10x^2-10x-120=0$;
$\therefore\ x^3-x^3-5x^2-5x^2+10x^2+4x-6x-10x=120$, &c.

28. $3x(x-2)(x+1)-3(x-1)(x+2)(x+1)+9(x+2)(x-2)=0$;
$i.e.\ 3x^3-3x^2-6x-3x^3-6x^2+3x+6+9x^2-36=0$, &c.

29. $3(x-1)^2+2(x+1)^2=5(x+1)(x-1)$;
$i.e.\ 3x^2-6x+3+2x^2+4x+2=5x^2-5$, &c.

30. $5(x-2)(x+3)-2(x-3)(x+2)=3(x+2)(x+3)$;
$i.e.\ 5x^2+5x-30-2x^2+2x+12=3x^2+15x+18$, &c.

31. $\dfrac{8x}{x^3-1}=\dfrac{(x+4)-(x-4)}{(x-4)(x+4)}$; $i.e.\ \dfrac{8x}{x^3-1}=\dfrac{8}{(x-4)(x+4)}$;
$\therefore\ x(x-4)(x+4)=x^3-1$; $i.e.\ x^3-16x=x^3-1$, &c.

32. $\dfrac{12x}{x^3-9}=\dfrac{2}{x-3}-\dfrac{2}{x+3}$; $\therefore\ \dfrac{12x}{x^3-9}=\dfrac{12}{(x-3)(x+3)}$;
$\therefore\ x(x^2-9)=x^3-9$, &c.

33. $\dfrac{4}{x^2-1}=\dfrac{1}{1-x}-\dfrac{1}{1+x}$; $\therefore\ \dfrac{4}{x^2-1}=\dfrac{2x}{(1-x)(1+x)}$, $i.e.\ \dfrac{4}{x^2-1}=\dfrac{-2x}{x^2-1}$;
$\therefore\ 4=-2x$, &c.

34. $\dfrac{3}{(x-3)(x+3)}+\dfrac{1}{x+3}=\dfrac{-2}{x-3}$; $\therefore\ 3+(x-3)=-2(x+3)$, &c.

35. $1+\dfrac{2}{2x-1}-\dfrac{8}{4x^2-1}=1-\dfrac{2}{2x+1}$; $\therefore\ 2(2x+1)-8=-2(2x-1)$, &c.

36. $1+\dfrac{6}{3x-1}+\dfrac{5}{1-9x^2}=1+\dfrac{7}{3x+1}$; $\therefore\ 6(3x+1)-5=7(3x-1)$, &c.

37. Multiply by $2(x-3)(x-5)$; then $2(x-3)+3(x-5)=10$, &c.

38. Multiply by $2(x-2)(x-5)$; then $(x-5)+6(x-2)=2(7x+1)$, &c.

39. Multiply by $(2-3x)(6-5x)(10+x)$; then

$$6(6-5x)(10+x)+10(2-3x)(10+x)+4(2-3x)(6-5x)=0;$$

i.e. $(360-264x-30x^2)+(200-280x-30x^2)+(48-112x+60x^2)=0$;

whence $$656x=608.$$

40. $6(3-7x)(5+x)+7(3x-1)(5+x)-(3x-1)(3-7x)=0$;

$\therefore\ (90-192x-42x^2)+(-35+98x+21x^2)-(-3+16x-21x^2)=0$, &c.

41. $$\frac{(x+6)-(x+5)}{(x+5)(x+6)}=\frac{(x+8)-(x+6)}{(x+6)(x+8)};\ i.e.\ \frac{1}{(x+5)(x+6)}=\frac{2}{(x+6)(x+8)};$$

$$\therefore\ x+8=2(x+5),\ \&c.$$

42. $$\frac{(x+1)-(x+7)}{(x+7)(x+1)}=\frac{(x+3)-(x+1)}{(x+1)(x+3)};\ i.e.\ \frac{-6}{(x+7)(x+1)}=\frac{2}{(x+1)(x+3)};$$

$$\therefore\ -3(x+3)=(x+7),\ \&c.$$

43. $$\frac{1}{x+2}-\frac{1}{x+4}=\frac{1}{x+8}-\frac{1}{x+10};$$

i.e. $$\frac{(x+4)-(x+2)}{(x+2)(x+4)}=\frac{(x+10)-(x+8)}{(x+8)(x+10)};\ \text{or}\ \frac{2}{(x+2)(x+4)}=\frac{2}{(x+8)(x+10)};$$

$$\therefore\ (x+2)(x+4)=(x+8)(x+10),\ \&c.$$

44. $$\frac{(x-6)-(x-2)}{(x-2)(x-6)}=\frac{(x-8)-(x-4)}{(x-4)(x-8)};\ i.e.\ \frac{-4}{(x-2)(x-6)}=\frac{-4}{(x-4)(x-8)};$$

$$\therefore\ (x-2)(x-6)=(x-4)(x-8),\ \&c.$$

45. $$\frac{1}{x-5}-\frac{1}{x-4}=\frac{1}{x+1}-\frac{1}{x+2};$$

i.e. $$\frac{(x-4)-(x-5)}{(x-5)(x-4)}=\frac{(x+2)-(x+1)}{(x+1)(x+2)};\ \text{or}\ \frac{1}{(x-5)(x-4)}=\frac{1}{(x+1)(x+2)};$$

thus $$(x-5)(x-4)=(x+1)(x+2),\ \&c.$$

46. $$\frac{x}{x-2}-\frac{x+1}{x-1}=\frac{x-8}{x-6}-\frac{x-9}{x-7};$$

$$i.e.\ \frac{x(x-1)-(x-2)(x+1)}{(x-2)(x-1)}=\frac{(x-8)(x-7)-(x-6)(x-9)}{(x-6)(x-7)};$$

or $$\frac{2}{(x-2)(x-1)}=\frac{2}{(x-6)(x-7)};\ \therefore\ (x-2)(x-1)=(x-6)(x-7),\ \&c.$$

47. $$2-\frac{1}{x+1}+2-\frac{1}{x+5}=2-\frac{1}{x+2}+2-\frac{1}{x+4};$$

$$\therefore\ \left(\frac{1}{x+2}-\frac{1}{x+1}\right)+\left(\frac{1}{x+4}-\frac{1}{x+5}\right)=0;$$

$$\therefore\ -\frac{1}{(x+1)(x+2)}+\frac{1}{(x+4)(x+5)}=0;$$

$$\therefore\ -(x+4)(x+5)+(x+1)(x+2)=0,\ \&c.$$

48. $1+\dfrac{1}{2x-4}-1-\dfrac{1}{2x-5}=1+\dfrac{1}{2x-8}-1-\dfrac{1}{2x-9};$

$$\therefore \frac{1}{2x-4}-\frac{1}{2x-5}=\frac{1}{2x-8}-\frac{1}{2x-9};$$

$$\therefore -\frac{1}{(2x-4)(2x-5)}=-\frac{1}{(2x-8)(2x-9)};$$

$$\therefore (2x-8)(2x-9)=(2x-4)(2x-5), \text{ \&c.}$$

49. $1+\dfrac{2}{x+5}+1+\dfrac{2}{x+7}=1+\dfrac{2}{x+4}+1+\dfrac{2}{x+8};$

$$\therefore \left(\frac{1}{x+5}-\frac{1}{x+4}\right)+\left(\frac{1}{x+7}-\frac{1}{x+8}\right)=0;$$

$$\therefore -\frac{1}{(x+4)(x+5)}+\frac{1}{(x+7)(x+8)}=0;$$

$$\therefore -(x+7)(x+8)+(x+4)(x+5)=0, \text{ \&c.}$$

50. $4-\dfrac{1}{4x-3}+5+\dfrac{2}{8x-9}=4-\dfrac{2}{8x-7}+5+\dfrac{1}{4x-5};$

$$\therefore \frac{2}{8x-9}-\frac{1}{4x-3}=\frac{1}{4x-5}-\frac{2}{8x-7};$$

$$\therefore \frac{3}{(8x-9)(4x-3)}=\frac{3}{(4x-5)(8x-7)};$$

$$\therefore (4x-5)(8x-7)=(4x-3)(8x-9), \text{ \&c.}$$

51. $1-\dfrac{2}{x-5}-1+\dfrac{2}{x-6}=1-\dfrac{2}{2x-5}-1+\dfrac{2}{2x-9};$

$$\therefore -\frac{1}{x-5}+\frac{1}{x-6}=-\frac{1}{2x-5}+\frac{1}{2x-9};$$

$$\therefore \frac{1}{(x-5)(x-6)}=\frac{4}{(2x-5)(2x-9)};$$

$$\therefore (2x-5)(2x-9)=4(x-5)(x-6), \text{ \&c.}$$

52. $\dfrac{7}{x-9}-\dfrac{7}{x+2}=\dfrac{11}{x-4}-\dfrac{11}{x+3};$ *i.e.*

$$\frac{7(x+2)-7(x-9)}{(x-9)(x+2)}=\frac{11(x+3)-11(x-4)}{(x-4)(x+3)}; \text{ or } \frac{77}{(x-9)(x+2)}=\frac{77}{(x-4)(x+3)};$$

$$\therefore (x-9)(x+2)=(x-4)(x+3), \text{ \&c.}$$

53. $\dfrac{16}{x-17}-\dfrac{16}{x+4}=\dfrac{21}{x-9}-\dfrac{21}{x+7};$

i.e. $\dfrac{16(x+4)-16(x-17)}{(x-17)(x+4)}=\dfrac{21(x+7)-21(x-9)}{(x-9)(x+7)};$

or $\dfrac{336}{(x-17)(x+4)}=\dfrac{336}{(x-9)(x+7)}$; $\therefore (x-17)(x+4)=(x-9)(x+7)$, &c.

54. $(x+a)(x-a)+(x+b)(x-b)=2(x-a)(x-b)$;

$$\therefore\ 2x^2-a^2-b^2=2x^2-2(a+b)x+2ab,$$

whence $$2(a+b)x=a^2+b^2+2ab, \text{ \&c.}$$

55. $$\frac{(x+a)(x+b)-(x-a)(x-b)}{(x-a)(x+b)}=\frac{2(a+b)}{x};$$

i.e. $$\frac{2(a+b)x}{(x-a)(x+b)}=\frac{2(a+b)}{x};\quad \therefore\ x^2=(x-a)(x+b), \text{ \&c.}$$

56. $ax(b+x)+bx(a+x)=(a+b)(a+x)(b+x)$;

$$\therefore\ (a+b)x^2+2abx=(a+b)\{x^2+(a+b)x+ab\};$$

whence $$-x(a^2+b^2)=ab(a+b), \text{ \&c.}$$

57. $$\frac{a+c}{x+2b}+\frac{b+c}{x+2a}=\frac{a+c}{x+a+b}+\frac{b+c}{x+a+b};$$

$$\therefore\ \frac{a+c}{x+2b}-\frac{a+c}{x+a+b}=\frac{b+c}{x+a+b}-\frac{b+c}{x+2a};$$

i.e. $$\frac{(a+c)(x+a+b)-(a+c)(x+2b)}{(x+2b)(x+a+b)}=\frac{(b+c)(x+2a)-(b+c)(x+a+b)}{(x+a+b)(x+2a)};$$

or $$\frac{(a+c)(a-b)}{(x+2b)(x+a+b)}=\frac{(b+c)(a-b)}{(x+a+b)(x+2a)};\quad \therefore\ \frac{a+c}{x+2b}=\frac{b+c}{x+2a};$$

thus $$(a+c)(x+2a)=(b+c)(x+2b);$$

whence $$(a-b)x=2b^2+2bc-2a^2-2ac=2(b-a)(a+b+c), \text{ \&c.}$$

58. $$1+\frac{a-b}{x-a}-1+\frac{a-b}{x-b}=\frac{2(a-b)}{x-a-b};$$

$$\therefore\ \frac{a-b}{x-a}+\frac{a-b}{x-b}=\frac{2(a-b)}{x-a-b};\quad \therefore\ \frac{1}{x-a}+\frac{1}{x-b}=\frac{2}{x-a-b};$$

$$\therefore\ (x-b)(x-a-b)+(x-a)(x-a-b)=2(x-a)(x-b), \text{ \&c.}$$

MISCELLANEOUS EXAMPLES. III. PAGES 151—154.

A. 1. $3x^2y-3xy^2+y^3-(x^3-3x^2y+3xy^2)=$ &c.

2. Both expressions are equal to $x^2y+x^2z+y^2z+y^2x+z^2x+z^2y+2xyz$.

3. (i) $x^4-y^4=(x^2-y^2)(x^2+y^2)=(x-y)(x+y)(x^2+y^2)$.

(ii) $(a^2-b^2)+(2a-2b)=(a-b)(a+b)+2(a-b)=(a-b)(a+b+2)$.

(iii) $x^2-3x-28=(x-7)(x+4)$.

4.

$$\begin{array}{l}
x^4+9x-20\;)\;5x^4+\;9x^3-\;64\;(\;5\\
\hphantom{x^4+9x-20\;)\;}\underline{5x^4+45x\;-100\quad}\\
\hphantom{x^4+9x-20\;)\;5x}9\;)\;\underline{9x^3-\;45x+36}\\
\hphantom{x^4+9x-20\;)\;5x^4+\;}x^3-\;\;5x+\;4\;)\;x^4+9x\;-20\;\;(\;x\\
\hphantom{x^4+9x-20\;)\;5x^4+\;x^3-\;\;5x+\;4\;)\;}\underline{x^4-5x^2+\;4x}\\
\hphantom{x^4+9x-20\;)\;5x^4+\;x^3-\;\;5x+\;4\;)\;x}5\;)\;\underline{5x^2+\;5x-20}\\
\hphantom{x^4+9x-20\;)\;5x^4+\;x^3-\;\;5x+\;4\;)\;x^4+}x^2+\;\;x-\;4\;)\;x^3-5x\;+4\;\;(\;x-\\
\hphantom{x^4+9x-20\;)\;5x^4+\;x^3-\;\;5x+\;4\;)\;x^4+x^2+\;\;x-\;4\;)\;}\underline{x^3+\;x^2-4x\quad}\\
\hphantom{x^4+9x-20\;)\;5x^4+\;x^3-\;\;5x+\;4\;)\;x^4+x^2+\;\;x-\;4\;)\;x^3}-\;x^2-\;x+4\\
\hphantom{x^4+9x-20\;)\;5x^4+\;x^3-\;\;5x+\;4\;)\;x^4+x^2+\;\;x-\;4\;)\;x^3}-\;x^2-\;x+4
\end{array}$$

$$\therefore\ \frac{x^4+9x-20}{5x^5+9x^4-64x}=\frac{(x^2+x-4)\,(x^2-x+5)}{x\,(x^2+x-4)\,(5x^2+4x+16)}=\&\text{c.}$$

5. (i) Multiply by 6; then $3\,(x-1)+2\,(2x-7)=6\,(x-2)$; whence $x=5$.

(ii) Since $3x-4y+2=5x-6y-2$; $\therefore\ -2x+2y=-4$, and since $5x-6y-2=7x+2y+4$, $\therefore\ -2x-8y=6$.

Whence, by subtraction $10y=-10$; thus $y=-1$, &c.

(iii) $\dfrac{a}{x+b}+\dfrac{b}{x+a}=\dfrac{a}{x}+\dfrac{b}{x}$; $\therefore\ \dfrac{a}{x+b}-\dfrac{a}{x}=\dfrac{b}{x}-\dfrac{b}{x+a}$;

or
$$\frac{ax-a\,(x+b)}{x\,(x+b)}=\frac{b\,(x+a)-bx}{x\,(x+a)};\ \textit{i.e.}\ \frac{-ab}{x\,(x+b)}=\frac{ab}{x\,(x+a)};$$

$$\therefore\ -\frac{1}{x+b}=\frac{1}{x+a};\ \text{or}\ -(x+a)=x+b;\ \&\text{c.}$$

6. Let A have x shillings, and B y shillings.

Then, from the question,

$$(x-10)=3\,(y+10)\ \text{ and }\ (x+5)=4\,(y-5).$$

Whence $$x-3y=40,\ \text{and}\ x-4y=-25,\ \&\text{c.}$$

B. 1. (i) $5\,(5+3)(5+3-6)-5\,(5-3)(5-3+6)=5\times8\times2-5\times2\times8=0$.

(ii) $(-3)(-3-2)(-3-2+4)-(-3)(-3+2)(-3+2-4)$
$=(-3)(-5)(-1)-(-3)(-1)(-5)=0$.

2. (i) Each expression $=4a^2-12ab+12b^2$.

(ii) $$\left(ab-\frac{1}{ab}\right)\left(\frac{a}{b}-\frac{b}{a}\right)=ab\times\frac{a}{b}-ab\times\frac{b}{a}-\frac{1}{ab}\times\frac{a}{b}+\frac{1}{ab}\times\frac{b}{a}$$

$$=a^2-b^2-\frac{1}{b^2}+\frac{1}{a^2}.$$

3. (i) $x\,(x^2-4x+4)=x\,(x-2)^2$.

(iii) $(x^3-x^2)+(x-1)=x^2\,(x-1)+(x-1)=(x-1)\,(x^2+1)$.

4. (i) $$\frac{1}{x-3}+\frac{1}{x+3}-\frac{6}{(x-3)(x+3)}=\frac{(x+3)+(x-3)-6}{(x-3)(x+3)}$$
$$=\frac{2x-6}{(x-3)(x+3)}=\frac{2(x-3)}{(x-3)(x+3)}=\frac{2}{x+3}.$$

(ii) $$\left(y-\frac{a^2-xy}{y-x}\right)\left(x+\frac{a^2-xy}{y-x}\right)+\left(\frac{a^2-xy}{y-x}\right)^2$$
$$=\frac{y(y-x)-(a^2-xy)}{y-x}\times\frac{x(y-x)+(a^2-xy)}{y-x}+\left(\frac{a^2-xy}{y-x}\right)^2$$
$$=\frac{y^2-a^2}{y-x}\times\frac{a^2-x^2}{y-x}+\left(\frac{a^2-xy}{y-x}\right)^2=\frac{(y^2-a^2)(a^2-x^2)+(a^2-xy)^2}{(y-x)^2}$$
$$=\frac{(-a^4+a^2x^2+a^2y^2-x^2y^2)+(a^4-2a^2xy+x^2y^2)}{(y-x)^2}$$
$$=\frac{a^2x^2-2a^2xy+a^2y^2}{x^2-2xy+y^2}=a^2.$$

5. (i) Multiply by $(x-1)(x-2)(x-3)$; then
$$(x-2)(x-3)+2(x-1)(x-3)=3(x-1)(x-2);$$
$$\therefore\ x^2-5x+6+2x^2-8x+6=3x^2-9x+6,\ \&\text{c}.$$

Or thus: $$\frac{1}{x-1}-\frac{1}{x-3}+\frac{2}{x-2}-\frac{2}{x-3}=0;$$
$$\therefore\ -\frac{2}{(x-1)(x-3)}-\frac{2}{(x-2)(x-3)}=0;\ \therefore\ x-2+x-1=0.$$

(ii) Multiply by a, b respectively; then $x-\frac{a}{b}y=a$, $x+\frac{b}{a}y=a$; whence
$$y=0,\ x=a.$$

6. Let $\frac{x}{y}$ be the fraction; then $\frac{x+2}{y}=\frac{1}{5}$ and $\frac{x}{y-2}=\frac{1}{6}$;
$$\therefore\ 5x-y=-10 \text{ and } 6x-y=-2,\ \&\text{c}.$$

C. 1. (i) $$(x^2-2x+1)(y^2+1)-(y^2-2y+1)(x^2+1)$$
$$=(x^2+1)(y^2+1)-2x(y^2+1)-(y^2+1)(x^2+1)+2y(x^2+1)$$
$$=2\{x^2y+y-xy^2-x\}=2\{xy(x-y)-(x-y)\}=2(xy-1)(x-y).$$

(ii) Each expression $=xy+xz+yz$.

2. Let x, y be any two numbers; then we have to prove the identity
$$xy=\frac{1}{4}\{(x+y)^2-(x-y)^2\}.$$

3. (i) $(1+21x)(1-3x)$. (ii) $3y(x^3-8y^3)=3y(x-2y)(x^2+2xy+4y^2)$.
(iii) $\{x^3+3x+3x^2+1\}\{x^3+3x-3x^2-1\}=(x+1)^3(x-1)^3$.

4. (i) Find H. C. F. of numerator and denominator.

$$\begin{array}{r}
x^2-10x+21\ )\ x^3-46x\ -\ 21\ \ (\,x+10 \\
x^3-10x^2+\ 21x \\
\hline
10x^2-\ 67x-\ 21 \\
10x^2-100x+210 \\
\hline
33\ )\ 33x-231 \\
\hline
x-7\ )\ x^2-10x+21\ (\,x-3 \\
x^2-\ 7x \\
\hline
-\ 3x+21 \\
-\ 3x+21
\end{array}$$

Whence we find $\dfrac{x^2-10x+21}{x^3-46x-21}=\dfrac{(x-7)(x-3)}{(x-7)(x^2+7x+3)}=$&c.

(ii) $\dfrac{(x-1)-(x-2)(x-4)+(x-3)^2}{(x-1)(x-2)(x-3)}$

$$=\frac{(x-1)-(x^2-6x+8)+(x^2-6x+9)}{(x-1)(x-2)(x-3)}=\text{\&c.}$$

5. (i) Multiply by 24.

$$\therefore\ 12(x-3)-6(x-1)+4(x+1)-3(x-3)=0;$$

$$\therefore\ 12x-6x+4x-3x=36-6-4-9,\ \text{\&c.}$$

(ii) Multiply the first equation by b, and subtract the second; then

$$x(ab-a^2)=bc-c^2;$$

$$\therefore\ x=\frac{bc-c^2}{ab-a^2}=\frac{c(b-c)}{a(b-a)}.\quad \text{Similarly } y=\frac{c(a-c)}{b(a-b)}.$$

6. Let A have £x and B have £y. Then $x+y=100$ and $\dfrac{x}{2}+\dfrac{2y}{3}=55$.

D. 1. $\sqrt{27-27+64}-(3+3-4)^2=\sqrt{64}-2^2=8-4=4$.

2. (i) $(x^2-9x+20)-3(x^2-3x+2)+3(x^2+3x+2)-(x^2+9x+20)=0$.

(ii) $1+n+\dfrac{1}{2}n(n+1)=(n+1)\left(1+\dfrac{1}{2}n\right)=\dfrac{1}{2}(n+1)(n+2)$;

then $\dfrac{1}{2}(n+1)(n+2)+\dfrac{1}{6}n(n+1)(n+2)=\dfrac{1}{6}(n+1)(n+2)(3+n)$.

3. (i) $x(x^2+xy-6y^2)=x(x+3y)(x-2y)$.

(ii) $x^2(x+a)-a^2(x+a)=(x^2-a^2)(x+a)=$&c.

(iii) $x^2(y^2-1)-(y^2-1)=(x^2-1)(y^2-1)=$&c.

4. (i) $$\frac{(a-x)^2+4ax+(a+x)^2}{(a+x)(a-x)}=\frac{(a^2-2ax+x^2)+4ax+(a^2+2ax+x^2)}{(a+x)(a-x)}$$
$$=\frac{2(a^2+2ax+x^2)}{(a+x)(a-x)}=\frac{2(a+x)^2}{(a+x)(a-x)}=\frac{2(a+x)}{a-x}.$$

(ii) $$\frac{(x-5)(x-6)-2(x-4)(x-6)+(x-4)(x-5)}{(x-4)(x-5)(x-6)}$$
$$=\frac{(x^2-11x+30)-(2x^2-20x+48)+(x^2-9x+20)}{(x-4)(x-5)(x-6)}=\frac{2}{(x-4)(x-5)(x-6)}.$$

5. (i) Multiply by 84.

$$21(3x+1)-168(6-x)=12(5x-4)-28(x-2);$$
$$\therefore\ 63x+21-1008+168x=60x-48-28x+56,\ \&\text{c.}$$

(ii) $3y+\frac{9}{x}=-6;\ y+\frac{5}{x}=8.$

Multiply the second by 3 and subtract the first;

then $$\frac{6}{x}=30;\ \text{whence } x=\frac{1}{5},\ \&\text{c.}$$

6. Let $3x$ be the number of leaps by the hare before she is caught; then the greyhound will take $2x$ leaps, and these $2x$ leaps of the greyhound are equal to $2x\times\frac{7}{3}$ leaps of the hare. Hence $2x\times\frac{7}{3}=3x+60$, whence $x=36$. Thus the greyhound takes 72 leaps.

E. 1. $$(m+n)(m+n-1)=m(\overline{m-1+n})+n(m+\overline{n-1})$$
$$=m(m-1)+2mn+n(n-1).$$
$$\therefore\ (m+n)(m+n-1)(m+n-2)$$
$$=\{m(m-1)+2mn+n(n-1)\}(m+n-2)$$
$$=m(m-1)(\overline{m-2}+n)+2mn(\overline{m-1}+\overline{n-1})+n(n-1)(m+\overline{n-2})$$
$$=m(m-1)(m-2)+m(m-1)n+2m(m-1)n+2mn(n-1)$$
$$+mn(n-1)+n(n-1)(n-2).$$

2. $$\begin{array}{rl}
x-y\;) & x^4-y^4\;(\;x^3+x^2y+xy^2+y^3\\
& \underline{x^4-x^3y}\\
& \quad x^3y-y^4\\
& \quad \underline{x^3y-x^2y^2}\\
& \qquad x^2y^2-y^4\\
& \qquad \underline{x^2y^2-xy^3}\\
& \qquad\quad xy^3-y^4\\
& \qquad\quad xy^3-y^4
\end{array}$$

From this we obtain, $(x+y)^4-(2z)^4$ divided by $(x+y)-(2z)$
$$=(x+y)^3+(x+y)^2(2z)+(x+y)(2z)^2+(2z)^3=\&\text{c.}$$

3. (ii) $$a(a^2-2bc-b^2-c^2)=a\{a^2-(b+c)^2\}$$
$$=a\{a-(b+c)\}\{a+(b+c)\}=\&\text{c.}$$

4. $8a^3-18ab^2=2a\,(4a^2-9b^2)=2a\,(2a-3b)(2a+3b)$,
$8a^3+8a^2b-6ab^2=2a\,(4a^2+4ab-3b^2)=2a\,(2a+3b)\,(2a-b)$,
$4a^2-8ab+3b^2=(2a-3b)\,(2a-b)$;
whence L. C. M. $=2a\,(2a-3b)\,(2a+3b)\,(2a-b)$.

5. $1+\dfrac{a}{x-a}=\dfrac{x}{x-a}$.

Then $\dfrac{x}{x-a}+\dfrac{bx}{(x-a)\,(x-b)}=\dfrac{x}{x-a}\left(1+\dfrac{b}{x-b}\right)=\dfrac{x^2}{(x-a)\,(x-b)}$.

Then $\dfrac{x^2}{(x-a)\,(x-b)}+\dfrac{cx^2}{(x-a)\,(x-b)\,(x-c)}$

$$=\frac{x^2}{(x-a)\,(x-b)}\left(1+\frac{c}{x-c}\right)=\frac{x^3}{(x-a)\,(x-b)\,(x-c)}.$$

6. Suppose that he owed £x. He was able to pay $\dfrac{1}{4}$ of his debt; thus his assets were £ $\dfrac{x}{4}$.

If his assets had been £ $\dfrac{5x}{4}$, and his debts £ $\dfrac{2x}{3}$, he would have had a balance in his favour of £140.

Hence $$\frac{5x}{4}-\frac{2x}{3}=140.$$

F. 1. Expression is equal to

$$\{-1-(-3+5)\}^2+\{-3-(-5+1)\}^2+\{-5-(-1+3)\}^2$$
$$=\{-1-2\}^2+\{-3+4\}^2+\{-5-2\}^2=3^2+1^2+(-7)^2=9+1+49=59.$$

2.
$$\begin{array}{r|l}
x^3+ax^2+a^2x+a^3\;) & x^8-2a^4x^4+a^8 \qquad (\,x^5-ax^4-a^4x+a^5 \\
 & x^8+a\,x^7+a^2x^6+a^3x^5 \\ \cline{2-2}
 & -a\,x^7-a^2x^6-a^3x^5-2a^4x^4+a^8 \\
 & -a\,x^7-a^2x^6-a^3x^5-\;\;a^4x^4 \\ \cline{2-2}
 & \qquad\qquad -a^4x^4+a^8 \\
 & \qquad\qquad -a^4x^4-a^5x^3-a^6x^2-a^7x \\ \cline{2-2}
 & \qquad\qquad\qquad a^5x^3+a^6x^2+a^7x+a^8 \\
 & \qquad\qquad\qquad a^5x^3+a^6x^2+a^7x+a^3
\end{array}$$

3. We have $(x-1)(x+1)$, $(x-1)^2$, $(x+1)^2$, $(x-1)\,(x^2+x+1)$,
and $(x+1)\,(x^2-x+1)$;
thus L. C. M. is $(x-1)^2\,(x+1)^2\,(x^2+x+1)\,(x^2-x+1)$.

4. (i) $\dfrac{1}{2x+y}+\dfrac{1}{2x-y}+\dfrac{3x}{y^2-4x^2}=\dfrac{(2x-y)+(2x+y)-3x}{(2x-y)\,(2x+y)}$

$$=\frac{x}{(2x-y)\,(2x+y)}=\&c.$$

(ii) The expression on the left is

$$\frac{(x-a)(y-a)(b-c)+(x-b)(y-b)(c-a)+(x-c)(y-c)(a-b)}{(a-b)(a-c)(b-c)}.$$

$$N=xy\,\{(b-c)+(c-a)+(a-b)\}-(x+y)\,\{a\,(b-c)+b\,(c-a)+c\,(a-b)\}$$
$$+a^2(b-c)+b^2(c-a)+c^2(a-b).$$

The coefficients of xy and $(x+y)$ are at once seen to be zero; also $a^2(b-c)+b^2(c-a)+c^2(a-b)$ will be found to be $(a-b)(b-c)(a-c)$.

5. (i) $\dfrac{1}{x+a}+\dfrac{1}{x+b}=\dfrac{1}{x+a+b}+\dfrac{1}{x+a+b}$.

$$\therefore \frac{1}{x+a}-\frac{1}{x+a+b}=\frac{1}{x+a+b}-\frac{1}{x+b};$$

$$i.e.\ \frac{(x+a+b)-(x+a)}{(x+a)(x+a+b)}=\frac{(x+b)-(x+a+b)}{(x+a+b)(x+b)};$$

or $$\frac{b}{(x+a)(x+a+b)}=\frac{-a}{(x+a+b)(x+b)};\ \therefore b\,(x+b)=-a\,(x+a),\ \&c.$$

(ii) Multiply the third equation by 5 and subtract from the first; then multiply the third equation by 3 and subtract from the second.

Then $$12y-2z=-12 \text{ and } 13y-4z=-13.$$

Hence $y=-1$, $z=0$. Then, on substitution, we find that $x=3$.

6. Suppose there are x acres of arable and y of pasture. Then

$$30x+40y=550\times 20, \text{ and } 20x+35y=387\tfrac{1}{2}\times 20.$$

XLIII. Pages 158, 159.

1. Either $x-1=0$, or $x+2=0$; that is $x=1$, or $x=-2$. Similarly for 2—16.

17. $x(x-1)=0$; whence $x=0$, or $x-1=0$, &c.

18. $x(x-2)=0$, &c.

19. $x(x+3)=0$, &c.

20. $x(2x-3)=0$, &c.

21. $x(2x-5)=0$, &c.

22. $x(3x+1)=0$, &c.

23. $x^2-5x=0$; *i.e.* $x(x-5)=0$, &c.

24. $2x^2-x=0$; *i.e.* $x(2x-1)=0$, &c.

25. $3x^2-5x=0$; *i.e.* $x(3x-5)=0$, &c.

26. $5x^2+6x=0$; *i.e.* $x(5x+6)=0$, &c.

27. $x^2-bx=0$; *i.e.* $x(x-b)=0$, &c.

28. $ax^2-bx=0$; *i.e.* $x(ax-b)=0$, &c.

29. $6x^2-x^2-5=0$; $\therefore 5x^2-5=0$; $\therefore x^2-1=0$; that is

$$(x-1)(x+1)=0,\ \&c.$$

30. $5x^2+25=3x^2+75$, whence $2x^2-50=0$; $\therefore\ x^2-25=0$;
i.e. $(x-5)(x+5)=0$, &c.

31. $5x^2+20=4x^2+36$; $\therefore\ x^2-16=0$; *i.e.* $(x-4)(x+4)=0$, &c.

32. $2x^2+14=7x^2+14$; $\therefore\ -5x^2=0$; $\therefore\ x=0$.

33. $5x^2+15-x^2+25=76$; $\therefore\ 4x^2-36=0$; $\therefore\ x^2-9=0$;
i.e. $(x-3)(x+3)=0$, &c.

34. $7x^2-7-x^2+9=56$; $\therefore\ 6x^2-54=0$; $\therefore\ x^2-9=0$;
i.e. $(x-3)(x+3)=0$, &c.

35. $3x^2+25x^2+20x+4=20x+32$; $\therefore\ 28x^2-28=0$; $\therefore\ x^2-1=0$;
i.e. $(x-1)(x+1)=0$, &c.

36. $17+3x=\frac{x^2}{2}+3x+\frac{9}{2}-28$; whence $x^2-81=0$;
i.e. $(x-9)(x+9)=0$, &c.

37. $(x-2)(x-3)=0$, &c.

38. $(x-3)(x-4)=0$, &c.

39. $(x-2)(x-10)=0$, &c.

40. $(x-4)(x-5)=0$, &c.

41. $(x-4)(x-7)=0$, &c.

42. $(x-10)(x-15)=0$, &c.

43. $(x+7)(x-12)=0$, &c.

44. $(x+12)(x-13)=0$, &c.

45. $(x+15)(x-10)=0$, &c.

46. $x^2+2x-3=0$; *i.e.* $(x+3)(x-1)=0$, &c.

47. $x^2+4x-45=0$; *i.e.* $(x+9)(x-5)=0$, &c.

48. $x^2-3x-10=0$; *i.e.* $(x-5)(x+2)=0$, &c.

49. $x(x^2+11x-180)=0$; *i.e.* $x(x+20)(x-9)=0$, &c.

50. $x^3-x^2-132x=0$; *i.e.* $x(x^2-x-132)=0$; or $x(x+11)(x-12)=0$, &c.

XLIV. PAGE 161.

1. $x^2-4x-45=0$; or $(x+5)(x-9)=0$, &c.

2. $x^2-6x-91=0$; or $(x+7)(x-13)=0$, &c.

3. $x^2-6x+9=x+3$; $\therefore\ x^2-7x+6=0$; or $(x-1)(x-6)=0$, &c.

4. $x^2-8x+16=x-2$; $\therefore\ x^2-9x+18=0$; or $(x-3)(x-6)=0$, &c.

5. $x^2-3x+2=20$; $\therefore\ x^2-3x-18=0$; or $(x-6)(x+3)=0$, &c.

6. $x^2+4x+3=2x+6$; $\therefore\ x^2+2x-3=0$; or $(x+3)(x-1)=0$, &c.

7. $4x+3=x^2+2x$; $\therefore\ x^2-2x-3=0$; or $(x-3)(x+1)=0$, &c.

8. $4x-3=2x-x^2$; $\therefore\ x^2+2x-3=0$; or $(x+3)(x-1)=0$, &c.

9. $x^3+3x^2+3x+1=x^3-3x^2+3x-1+26$; $\therefore\ 6x^2-24=0$;
$\therefore\ x^2-4=0$; or $(x-2)(x+2)=0$, &c.

10. $x^3-3x^2+3x-1=x^3+3x^2+3x+1-56$; $\therefore\ -6x^2+54=0$;
$\therefore\ x^2-9=0$; or $(x-3)(x+3)=0$, &c.

11. $(3x+1)^2=0$; the only solution of which is $3x+1=0$; or $x=-\frac{1}{3}$.

12. $9x^2-24x+16=0$; *i.e.* $(3x-4)^2=0$; $\therefore\ 3x-4=0$, &c.

13. $x^2-53x+150=0$; *i.e.* $(x-3)(x-50)=0$, &c.

14. $x^2-2x-99=0$; *i.e.* $(x-11)(x+9)=0$, &c.

15. $3x^2-10x+3=0$; *i.e.* $(3x-1)(x-3)=0$, &c.

16. $3x^2+11x-20=0$; *i.e.* $(3x-4)(x+5)=0$, &c.

17. $4x^2+21x-18=0$; *i.e.* $(4x-3)(x+6)=0$, &c.
Or if the student cannot see the factors by inspection, he may proceed,

$$x^2+\frac{21}{4}x-\frac{9}{2}=0;\ \therefore\ x^2+\frac{21}{4}x+\left(\frac{21}{8}\right)^2-\left\{\frac{9}{2}+\left(\frac{21}{8}\right)^2\right\}=0,$$

$$i.e.\ \left(x+\frac{21}{8}\right)^2-\frac{729}{64}=0;\ \text{or}\ \left(x+\frac{21}{8}\right)^2-\left(\frac{27}{8}\right)^2=0;$$

or, $$\left\{\left(x+\frac{21}{8}\right)-\frac{27}{8}\right\}\left\{\left(x+\frac{21}{8}\right)+\frac{27}{8}\right\}=0,\ \text{\&c.}$$

Similarly with the others.

18. $6x^2+55x-50=0$; *i.e.* $(6x-5)(x+10)=0$, &c.

19. $24x^2-30x-75=0$; *i.e.* $(12x+15)(2x-5)=0$, &c.

20. $x^2-35x-200=0$; *i.e.* $(x+5)(x-40)=0$, &c.

21. $19x^2-39x+2=0$; *i.e.* $(19x-1)(x-2)=0$, &c.

22. $17x^2-70x+8=0$; *i.e.* $(17x-2)(x-4)=0$, &c.

23. $51x^2+19x-10=0$; *i.e.* $(17x-5)(3x+2)=0$, &c.

24. $(11x-1)(10x-1)=0$, &c.

25. $21x^2-13x-20=0$; *i.e.* $(3x-4)(7x+5)=0$, &c.

26. $21x^2+23x-20=0$; *i.e.* $(3x+5)(7x-4)=0$, &c.

27. $6x^2-13x+6=0$; *i.e.* $(2x-3)(3x-2)=0$, &c.

28. $6x^2-5x-1=0$; *i.e.* $(6x+1)(x-1)=0$, &c.

29. $(3x-4)(3x-17)=0$, &c.

30. $16x^2-16x+3=0$; *i.e.* $(4x-3)(4x-1)=0$, &c.

31. $(29x+46)(x-3)=0$, &c.

32. $29x^2+11x-138=0$; *i.e.* $(29x+69)(x-2)=0$, &c.

33. $x^2+10x-231=0$; *i.e.* $(x+21)(x-11)=0$, &c.

34. $(x+2)^2-4(x-1)^2=0$; *i.e.* $\{(x+2)-2(x-1)\}\{(x+2)+2(x-1)\}=0$;
thus either $(x+2)-2(x-1)=0$, whence $x=4$;
or $(x+2)+2(x-1)=0$, whence $x=0$.

35. $(x+6)^2-16(x-6)^2=0$; *i.e.* $\{(x+6)-4(x-6)\}\{(x+6)+4(x-6)\}=0$;
thus either $(x+6)-4(x-6)=0$, whence $x=10$;
or $(x+6)+4(x-6)=0$, whence $x=\frac{18}{5}$.

36. $(x+8)^2-9x^2=0$; *i.e.* $\{(x+8)-3x\}\{(x+8)+3x\}=0$;
thus either $(x+8)-3x=0$, whence $x=4$;
or $(x+8)+3x=0$, whence $x=-2$.

37. $(x-7)^2-49(x+2)^2=0$; *i.e.* $\{(x-7)-7(x+2)\}\{(x-7)+7(x+2)\}=0$;
thus either $(x-7)-7(x+2)=0$, whence $x=-\frac{7}{2}$;
or $(x-7)+7(x+2)=0$, whence $x=-\frac{7}{8}$.

38. $(x-a)(x-2a)=0$, &c.

39. $x^2-4ax+3a^2=0$; *i.e.* $(x-a)(x-3a)=0$, &c.

40. $x^2-2ax+2ab-b^2=0$; the term required to make x^2-2ax a perfect square is $+a^2$; thus we write $x^2-2ax+a^2-a^2+2ab-b^2=0$;
or $(x-a)^2-(a-b)^2=0$; *i.e.* $\{(x-a)-(a-b)\}\{(x-a)+(a-b)\}=0$;
thus either $(x-a)-(a-b)=0$, whence $x=2a-b$;
or $(x-a)+(a-b)=0$, whence $x=b$.

41. $4x^2+4ax+a^2-b^2=0$; *i.e.* $(2x+a)^2-b^2=0$;
i.e. $\{(2x+a)-b\}\{(2x+a)+b\}=0$;
thus either $(2x+a)-b=0$; or $(2x+a)+b=0$, &c.

42. $x^2+2ax-b^2-2ab=0$; the term required to make x^2+2ax a perfect square is $+a^2$; thus we write,

$$x^2+2ax+a^2-a^2-b^2-2ab=0;\ i.e.\ (x+a)^2-(a+b)^2=0;$$

$$i.e.\ \{(x+a)-(a+b)\}\{(x+a)+(a+b)\}=0;$$

thus either $(x+a)-(a+b)=0$, or $(x+a)+(a+b)=0$, &c.

XLV. Pages 165—167.

1. Multiply by the L. C. M. of denominators, *i.e.* by $2x$; then
$2x^2+2=5x$; $\therefore$ $2x^2-5x+2=0$; *i.e.* $(2x-1)(x-2)=0$, &c.

2. Multiply by $6x$, the L. C. M. of denominators; then
$6x^2+6=13x$; $\therefore$ $6x^2-13x+6=0$; *i.e.* $(2x-3)(3x-2)=0$, &c.

3. Multiply by $6x$, the L. C. M. of denominators; then
$3x^2+12=20x$; $\therefore$ $3x^2-20x+12=0$; *i.e.* $(3x-2)(x-6)=0$, &c.

4. Multiply by $4(x+5)$; then
$(x+5)^2-16=6(x+5)$; $\therefore$ $x^2+4x-21=0$; *i.e.* $(x-3)(x+7)=0$, &c.

5. Multiply by $6x(x-1)$; then $6x^2+6(x-1)^2=13x(x-1)$;
$\therefore$ $x^2-x-6=0$; *i.e.* $(x-3)(x+2)=0$, &c.

6. Multiply by $10(x+1)(x+2)$.
$10(x+1)^2+10(x+2)^2=29(x+1)(x+2)$;
$\therefore$ $9x^2+27x+8=0$; *i.e.* $(3x+1)(3x+8)=0$, &c.

7. Multiply by $5(x-2)(x+2)$; then
$5(x+2)-10(x-2)=3(x-2)(x+2)$; whence $3x^2+5x-42=0$;
i.e. $(3x+14)(x-3)=0$, &c.

8. Multiply by $3(x-4)(x+4)$; then
$21(x-4)-3(x+4)=5(x-4)(x+4)$; whence $5x^2-18x+16=0$;
i.e. $(5x-8)(x-2)=0$, &c.

9. Multiply by $(x+1)(x-2)$; then
$x(x+1)+4(x-2)=3(x-2)(x+1)$; whence $x^2-4x+1=0$;
$\therefore$ $(x-2)^2-3=0$; *i.e.* $(x-2+\sqrt{3})(x-2-\sqrt{3})=0$, &c.

10. Multiply by $3(x-1)(x+1)(x+10)$; then
$15(x-1)(x+10)-30(x-1)(x+1)=2(x+1)(x+10)$;
$\therefore$ $15x^2+135x-150-30x^2+30=2x^2+22x+20$;
$\therefore$ $17x^2-113x+140=0$; *i.e.* $(17x-28)(x-5)=0$, &c.

11. Multiply by $2(3+x)(2+3x)$; then
$(3+x)(2+3x)+2(2+3x)+2(3+x)=0$;
$\therefore$ $6+11x+3x^2+4+6x+6+2x=0$; *i.e.* $3x^2+19x+16=0$;
$\therefore$ $(3x+16)(x+1)=0$, &c.

12. Multiply by $3(x+3)(x+11)$; then
$(x+3)(x+11)+3(x+11)+3(x+3)=0$;
$\therefore$ $x^2+20x+75=0$; *i.e.* $(x+5)(x+15)=0$, &c.

13. Multiply by $(x-1)(x+1)$; then
$3(x-1)^2-2(x+1)^2=5(x-1)(x+1)$;
whence $2x^2+5x-3=0$; *i.e.* $(2x-1)(x+3)=0$, &c.

14. Multiply by $(x-1)(x-3)(x+3)$; then

$$3(x-3)(x+3)+4(x-1)(x+3)=15(x-1)(x-3);$$

whence $2x^2-17x+21=0$; *i.e.* $(2x-3)(x-7)=0$, &c.

15. Multiply by $18(5x+3)$; then

$$2(4x+3)(5x+3)=(6x-13)(5x+3)-18(7x-46);$$

whence $10x^2+227x-771=0$; *i.e.* $(10x+257)(x-3)=0$, &c.

16. Multiply by $12(x-1)$; then

$$3(x-1)(2x-3)=(x-1)(x-5)+12(5x-16);$$

whence $5x^2-69x+196=0$; *i.e.* $(5x-49)(x-4)=0$, &c.

17. Multiply by $11(2x+1)(3x-5)$; then

$$11(2x-1)(3x-5)+13(2x+1)(3x-5)=11(2x+1)(3x+5);$$

$\therefore\ 78x^2-377x-65=0$; $\therefore\ 6x^2-29x-5=0$; *i.e.* $(6x+1)(x-5)=0$, &c.

18. Multiply by $4(2x-3)(3x-2)$; then

$$4(2x-2)(3x-2)+4(2x-3)(3-3x)=5(3x-2);$$

$\therefore\ 24x^2-40x+16-24x^2+60x-36=15x-10$; $\therefore\ 5x-10=0$; $\therefore\ x=2$.

19. Multiply by $(x-2)(x+3)$; then

$3x(x+3)-4(x-2)-4(x+3)=0$; $\therefore\ 3x^2+9x-4x+8-4x-12=0$;

$\therefore\ 3x^2+x-4=0$; *i.e.* $(3x+4)(x-1)=0$, &c.

20. Multiply by $(x-3)(x+2)$; then

$5x(x+2)-6(x-3)-19(x+2)=0$; $\therefore\ 5x^2+10x-6x+18-19x-38=0$;

$\therefore\ x^2-3x-4=0$; *i.e.* $(x-4)(x+1)=0$, &c.

21. Multiply by $(x+1)(x-1)$; then $(x-1)^2+(x+1)^2=5x$;

$\therefore\ 2x^2-5x+2=0$; *i.e.* $(2x-1)(x-2)=0$, &c.

22. Multiply by $2x^2(x-3)(x+4)$; then

$$2x(x+4)+2x(x-3)=9(x-3)(x+4);$$

$\therefore\ 2x^2+8x+2x^2-6x=9x^2+9x-108$; $\therefore\ 5x^2+7x-108=0$;

i.e. $(5x+27)(x-4)=0$, &c.

23. Multiply by $16x(x-3)(x+3)$; then

$16(x+3)+16x=13(x-3)(x+3)$; $\therefore\ 16x+48+16x=13x^2-117$;

$\therefore\ 13x^2-32x-165=0$; *i.e.* $(13x+33)(x-5)=0$, &c.

24. Multiply by $72(x-2)(x+2)$; then $24(x+2)-360+7x(x-2)=0$;

whence $7x^2+10x-312=0$; *i.e.* $(7x+52)(x-6)=0$, &c.

25. Multiply by $8(x-1)(x+1)$; then

$$8+8(x+1)=7(x-1)(x+1)-8(x-1);$$

whence $7x^2-16x-15=0$; *i.e.* $(7x+5)(x-3)=0$, &c.

26. Multiply by $3(x-2)(x+2)$; then

$$3+9(x+2)=3(x-2)(x+2)+(x-2);$$

whence $-3x^2+8x+35=0$; *i.e.* $-(3x+7)(x-5)=0$, &c.

27. We have $\frac{1}{x}-\frac{1}{x+1}=\frac{1}{x+2}-\frac{1}{x+4}$; $\therefore \frac{1}{x(x+1)}=\frac{2}{(x+2)(x+4)}$.

Now multiply by $x(x+1)(x+2)(x+4)$; then $(x+2)(x+4)=2x(x+1)$;

whence $x^2-4x-8=0$; $\therefore (x-2)^2-12=0$;

i.e. $(x-2+\sqrt{12})(x-2-\sqrt{12})=0$.

28. We have $\frac{x}{x+1}-\frac{x-1}{x}=\frac{x-2}{x-1}-\frac{x+1}{x+2}$; $\therefore \frac{1}{x(x+1)}=\frac{-3}{(x-1)(x+2)}$.

Now multiply by $x(x-1)(x+1)(x+2)$; $\therefore (x-1)(x+2)=-3x(x+1)$;

whence $4x^2+4x-2=0$; or $x^2+x-\frac{1}{2}=0$;

$$\therefore \left(x+\frac{1}{2}\right)^2-\frac{3}{4}=0;\ \therefore \left\{\left(x+\frac{1}{2}\right)-\frac{\sqrt{3}}{2}\right\}\left\{\left(x+\frac{1}{2}\right)+\frac{\sqrt{3}}{2}\right\}=0,\ \text{\&c.}$$

29. $\frac{x}{x-3}-\frac{x+3}{x}+\frac{x}{x+3}-\frac{x-3}{x}=\frac{2}{3}$; $\therefore \frac{9}{x(x-3)}+\frac{9}{x(x+3)}=\frac{2}{3}$;

$$\therefore 27(x+3)+27(x-3)=2x(x+3)(x-3);$$

$$\therefore 27x=x(x^2-9);\ i.e.\ x(x^2-36)=0.$$

Hence $x=\pm 6$, the value $x=0$ being inadmissible [See Art. 148].

30. Re-arranging, we have $\frac{x+3}{x+1}-\frac{x+4}{x+2}=\frac{x-5}{x-3}-\frac{x-6}{x-4}$;

$\therefore \frac{2}{(x+1)(x+2)}=\frac{2}{(x-3)(x-4)}$; $\therefore (x+1)(x+2)=(x-3)(x-4)$;

whence $10x=10$, &c.

31. We have $\frac{-1}{(x-4)(x-5)}=\frac{-1}{(x-7)(x-8)}$;

$\therefore (x-4)(x-5)=(x-7)(x-8)$; whence $6x=36$, &c.

32. $x^2-3ax+2a^2=0$; *i.e.* $(x-a)(x-2a)=0$, &c.

33. $9x^2-6ax-a^2+b^2=0$; $\therefore (3x-a)^2-(2a^2-b^2)=0$;

$\therefore \{(3x-a)-\sqrt{(2a^2-b^2)}\}\{(3x-a)+\sqrt{(2a^2-b^2)}\}=0$, &c.

34. $ax^2+x(a^2-1)-a=0$; $\therefore (ax-1)(x+a)=0$, &c.

35. $ax^2-x(a^2+1)+a=0$; $\therefore (ax-1)(x-a)=0$, &c.

36. $x^2-2(a-b)x+b^2-2ab=0$;

$$\therefore x^2-2(a-b)x+(a-b)^2-(a-b)^2+b^2-2ab=0;$$

i.e. $\{x-(a-b)\}^2-a^2=0$; $\therefore \{x-(a-b)+a\}\{x-(a-b)-a\}=0$, &c.

37. $x^2+2(b-c)x+c^2-2bc=0$;

$\therefore\ x^2+2(b-c)x+(b-c)^2-(b-c)^2+c^2-2bc=0$; $\therefore\ \{x+(b-c)\}^2-b^2=0$;

$\therefore\ [x+(b-c)-b][x+(b-c)+b]=0$; *i.e.* $[x-c][x+2b-c]=0$, &c.

NOTE. In the following examples, wherever the student cannot see the factors by inspection, he should proceed as in Art. 143, or Art. 144. A root is often obvious by inspection; and in this case it is often well to proceed as in Art. 155.

38. $(b-c)x^2-\{(b-c)+(a-b)\}x+(a-b)=0$;
i.e. $\{(b-c)x-(a-b)\}(x-1)=0$, &c.

39. We have $(a+b)(x^2-1)+c(x-1)=0$;
i.e. $(x-1)\{(a+b)(x+1)+c\}=0$, &c.

40. $(ax-b)(bx-a)=0$, &c. **41.** $(bx+c)(cx+b)=0$, &c.

42. $(a^2-b^2)x^2-4abx-(a^2-b^2)=0$;
i.e. $\{(a-b)x-(a+b)\}\{(a+b)x+(a-b)\}=0$, &c.

43. $(b^2-a^2)x^2-2(b^2+a^2)x+(b^2-a^2)=0$;
i.e. $\{(b-a)x-(b+a)\}\{(b+a)x-(b-a)\}=0$, &c.

44. We have $x-a+\frac{1}{a}-\frac{1}{x}=0$; $\therefore\ x-a+\frac{1}{ax}(x-a)=0$;

$$\therefore\ (x-a)(ax+1)=0, \text{ \&c.}$$

45. Multiply by $(a+x)(a-x)$; then

$a(a-x)+a(a+x)=4(a+x)(a-x)$; whence $x^2-\frac{1}{2}a^2=0$;

$$\therefore\ \left(x-\frac{1}{\surd 2}a\right)\left(x+\frac{1}{\surd 2}a\right)=0, \text{ \&c.}$$

46. $\left(\frac{1}{x-a}-\frac{1}{b}\right)+\left(\frac{1}{x-b}-\frac{1}{a}\right)=0$; $\therefore\ \frac{a+b-x}{b(x-a)}+\frac{a+b-x}{a(x-b)}=0.$

Hence $a+b-x=0$ ....(I),

or $\frac{1}{b(x-a)}+\frac{1}{a(x-b)}=0$..............................(II).

From II, $a(x-b)+b(x-a)=0$, whence $x=2ab\div(a+b)$.

47. $\frac{a}{x-a}-\frac{a}{b}+\frac{b}{x-b}-\frac{b}{a}=0$; $\therefore\ \frac{a(a+b-x)}{b(x-a)}+\frac{b(a+b-x)}{a(x-b)}=0.$

Hence $a+b-x=0$ ....(I),

or $\frac{a}{b(x-a)}+\frac{b}{a(x-b)}=0$ ....(II).

From II, $a^2(x-b)+b^2(x-a)=0$, &c.

48. $\dfrac{a^2}{x-b} - a + \dfrac{b^2}{x-a} - b = 0$; $\therefore \dfrac{a(a+b-x)}{x-b} + \dfrac{b(a+b-x)}{x-a} = 0.$

Hence $$a+b-x=0 \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\text{(I)},$$

or $$\frac{a}{x-b} + \frac{b}{x-a} = 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\text{(II)}.$$

From II, $(a+b)x - a^2 - b^2 = 0$, &c.

49. Multiply by abx; then $bx^2 + a^2b = b^2x + a^2x$,

or $$bx^2 - (a^2+b^2)x + a^2b = 0; \quad \therefore (bx-a^2)(x-b) = 0, \text{ \&c.}$$

50. Multiply by $x(x+a+b)$; then $x = x+a+b+x(x+a+b)\left(\dfrac{1}{a}+\dfrac{1}{b}\right)$;

$$\therefore 0 = a+b+x(x+a+b)\frac{a+b}{ab}.$$

Now multiply by $\dfrac{ab}{a+b}$; then

$$0 = ab + x(x+a+b), \text{ i.e. } 0 = (a+x)(b+x), \text{ \&c.}$$

51. $\dfrac{1}{x-a-b} - \dfrac{1}{x} + \dfrac{1}{a} + \dfrac{1}{b} = 0$; $\therefore \dfrac{a+b}{x(x-a-b)} + \dfrac{a+b}{ab} = 0$;

$$\therefore ab + x(x-a-b) = 0; \text{ i.e. } (a-x)(b-x) = 0, \text{ \&c.}$$

52. $\left(\dfrac{1}{x+a} - \dfrac{1}{c+a}\right) + \left(\dfrac{1}{x+b} - \dfrac{1}{c+b}\right) = 0;$

$$\therefore \frac{c-x}{(c+a)(x+a)} + \frac{c-x}{(c+b)(x+b)} = 0.$$

Hence $$c-x=0 \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\text{(I)},$$

or $$\frac{1}{(c+a)(x+a)} + \frac{1}{(c+b)(x+b)} = 0 \ldots\ldots\ldots\ldots\ldots\ldots\text{(II)}.$$

From II, we have $(c+b)(x+b) + (c+a)(x+a) = 0$, whence

$$x = -\{a(a+c) + b(b+c)\} \div (a+b+2c).$$

53. $\left(\dfrac{x}{x+a} - \dfrac{c}{c+a}\right) + \left(\dfrac{x}{x+b} - \dfrac{c}{c+b}\right) = 0;$

$$\therefore \frac{a(x-c)}{(x+a)(c+a)} + \frac{b(x-c)}{(x+b)(c+b)} = 0.$$

Hence $$x-c=0 \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\text{(I)},$$

or $$\frac{a}{(x+a)(c+a)} + \frac{b}{(x+b)(c+b)} = 0 \ldots\ldots\ldots\ldots\ldots\ldots\text{(II)}.$$

From II, $a(x+b)(c+b) + b(x+a)(c+a) = 0$,

whence $$x = -\frac{ab(a+b+2c)}{ac+bc+2ab}.$$

54. $\dfrac{x+a}{x-a} - \dfrac{x-a}{x+a} + \dfrac{x+b}{x-b} - \dfrac{x-b}{x+b} = 0$; $\therefore \dfrac{4ax}{x^2-a^2} + \dfrac{4bx}{x^2-b^2} = 0$.

Hence $x = 0$ .... (I),

or $\dfrac{4a}{x^2-a^2} + \dfrac{4b}{x^2-b^2} = 0$ ....(II).

From (II), $a(x^2-b^2) + b(x^2-a^2) = 0$, whence $x^2 - ab = 0$, &c.

55. $\left(\dfrac{a}{x+a} - \dfrac{a-c}{x+a-c}\right) + \left(\dfrac{b}{x+b} - \dfrac{b+c}{x+b+c}\right) = 0$;

$$\therefore \frac{cx}{(x+a)(x+a-c)} - \frac{cx}{(x+b)(x+b+c)} = 0;$$

$$\therefore cx\{(x+b)(x+b+c) - (x+a)(x+a-c)\} = 0;$$

$$\therefore cx\{2x(b+c-a) + b^2 - a^2 + bc + ac\} = 0, \text{ \&c.}$$

XLVI. Pages 169, 170.

Square both sides of the equations 1 to 9; then

1. $x - 3 = 9$, &c.

2. $x - 7 = 16$, &c.

3. $3x + 1 = 16$, &c.

4. $25(x+7) = 16(3x-2)$, &c.

5. $9(x+3) = 4(3x+6)$, &c.

6. $(x+5) = 4(3x+4)$, &c.

7. $9(x+7) = 25(3x-2)$; whence $-66x = -113$, &c.

8. $x + 2 = x^2$; whence $x^2 - x - 2 = 0$; *i.e.* $(x-2)(x+1) = 0$, &c.

9. $x + 20 = x^2$; whence $x^2 - x - 20 = 0$; *i.e.* $(x-5)(x+4) = 0$, &c.

10. $\sqrt{(x+1)} = 5 - x$; $\therefore$ squaring, $x + 1 = 25 - 10x + x^2$;
whence $x^2 - 11x + 24 = 0$; *i.e.* $(x-3)(x-8) = 0$, &c.

11. $-\sqrt{(x+2)} = 4 - x$; $\therefore x + 2 = 16 - 8x + x^2$;
whence $x^2 - 9x + 14 = 0$; *i.e.* $(x-2)(x-7) = 0$, &c.

12. $3\sqrt{(x-2)} = 2 - x$; $\therefore 9(x-2) = 4 - 4x + x^2$;
whence $x^2 - 13x + 22 = 0$; *i.e.* $(x-2)(x-11) = 0$, &c.

13. $2\sqrt{(x-5)} = 5 - x$; $\therefore 4(x-5) = 25 - 10x + x^2$;
whence $x^2 - 14x + 45 = 0$; *i.e.* $(x-5)(x-9) = 0$, &c.

14. $9 + 4x = 4x^2 - 12x + 9$; whence $4x^2 - 16x = 0$; *i.e.* $4x(x-4) = 0$, &c.

15. $3x - 5 = \sqrt{(30x-71)}$; $\therefore 9x^2 - 30x + 25 = (30x - 71)$;
whence $9x^2 - 60x + 96 = 0$; $\therefore (3x-8)(x-4) = 0$, &c.

16. $-5\sqrt{x} = 3 - 2x$; $\therefore 25x = 9 - 12x + 4x^2$;
whence $4x^2 - 37x + 9 = 0$; *i.e.* $(4x-1)(x-9) = 0$, &c.

17. $\sqrt{(x+3)}=3-x$; $\therefore$ $x+3=9-6x+x^2$; whence $x^2-7x+6=0$;
i.e. $(x-1)(x-6)=0$, &c.

18. $4x^2+4x+1=6x+3$; whence $4x^2-2x-2=0$; $\therefore$ $(2x+1)(x-1)=0$, &c.

19. $7x-33=\sqrt{(3x-11)}$; $\therefore$ $49x^2-462x+1089=3x-11$;
whence $49x^2-465x+1100=0$; *i.e.* $(49x-220)(x-5)=0$, &c.

20. We have $\sqrt{(x+10)}=1-\sqrt{(x+1)}$; $\therefore$ $x+10=1+x+1-2\sqrt{(x+1)}$.
Hence $8=-2\sqrt{(x+1)}$; $\therefore 16=x+1$, &c.

21. Square both sides; then
$x+2\sqrt{x}\sqrt{(5x+1)}+5x+1=25$; $\therefore$ $2\sqrt{x}\sqrt{(5x+1)}=24-6x$;
$\therefore$ $\sqrt{x}\sqrt{(5x+1)}=12-3x$; $\therefore$ $x(5x+1)=144-72x+9x^2$;
whence $4x^2-73x+144=0$; *i.e.* $(4x-9)(x-16)=0$, &c.

22. $x+2\sqrt{x}\sqrt{(5x+1)}+5x+1=4$;
whence $2\sqrt{x}\sqrt{(5x+1)}=3-6x$; $\therefore$ $4x(5x+1)=9-36x+36x^2$;
whence $16x^2-40x+9=0$; *i.e.* $(4x-9)(4x-1)=0$, &c.

23. $2x+9-2\sqrt{(2x+9)}\sqrt{(x+4)}+x+4=1$;
whence $-2\sqrt{(2x+9)}\sqrt{(x+4)}=-12-3x$;
$\therefore$ $4(2x+9)(x+4)=144+72x+9x^2$;
whence $x^2+4x=0$; *i.e.* $x(x+4)=0$.

24. $7x+1-2\sqrt{(7x+1)}\sqrt{(3x+10)}+3x+10=1$;
whence $-2\sqrt{(7x+1)}\sqrt{(3x+10)}=-10-10x$; $\therefore$ $\sqrt{(7x+1)}\sqrt{(3x+10)}=5+5x$;
$\therefore$ $(7x+1)(3x+10)=25+50x+25x^2$;
whence $4x^2-23x+15=0$; *i.e.* $(4x-3)(x-5)=0$, &c.

25. $2x+11-2\sqrt{(2x+11)}\sqrt{(2x-5)}+2x-5=4$;
whence $-2\sqrt{(2x+11)}\sqrt{(2x-5)}=-2-4x$; $\therefore$ $\sqrt{(2x+11)}\sqrt{(2x-5)}=1+2x$;
$\therefore$ $(2x+11)(2x-5)=1+4x+4x^2$; whence $x=7$.

26. $4x+1-2\sqrt{(4x+1)}\sqrt{(x+3)}+x+3=4$;
$\therefore$ $-2\sqrt{(4x+1)}\sqrt{(x+3)}=-5x$; $\therefore$ $4(4x+1)(x+3)=25x^2$;
whence $9x^2-52x-12=0$; *i.e.* $(9x+2)(x-6)=0$, &c.

27. $8x+5-4\sqrt{(2x+1)}\sqrt{(8x+5)}+4(2x+1)=1$;
whence $-4\sqrt{(2x+1)}\sqrt{(8x+5)}=-8-16x$;
$\therefore$ $\sqrt{(2x+1)}\sqrt{(8x+5)}=2+4x$; $\therefore$ $(2x+1)(8x+5)=4+16x+16x^2$;
whence $2x+1=0$, &c.

28. $\sqrt{(x+4)}+\sqrt{(x+20)}=2\sqrt{(x+11)}$;
$\therefore$ $x+4+2\sqrt{(x+4)}\sqrt{(x+20)}+x+20=4(x+11)$;
$\therefore$ $2\sqrt{(x+4)}\sqrt{(x+20)}=2x+20$; $\therefore$ $\sqrt{(x+4)}\sqrt{(x+20)}=x+10$;
$\therefore$ $(x+4)(x+20)=x^2+20x+100$; whence $4x-20=0$, &c.

29. $4x+1-2\sqrt{(4x+1)}\sqrt{(x+3)}+x+3=x-2;$
whence $-2\sqrt{(4x+1)}\sqrt{(x+3)}=-4x-6;\ \therefore\ \sqrt{(4x+1)}\sqrt{(x+3)}=2x+3;$
$\therefore\ (4x+1)(x+3)=4x^2+12x+9;$ whence $x-6=0.$

30. $2x+4+2\sqrt{(2x+4)}\sqrt{(3x+7)}+3x+7=12x+9;$
whence $2\sqrt{(2x+4)}\sqrt{(3x+7)}=7x-2;\ \therefore\ 4(2x+4)(3x+7)=49x^2-28x+4;$
whence $25x^2-132x-108=0;$ *i.e.* $(25x+18)(x-6)=0.$

31. $6x+1+2\sqrt{(6x+1)}\sqrt{2(1-x)}+2(1-x)=7x+6;$
whence $2\sqrt{(6x+1)}\sqrt{2(1-x)}=3x+3;\ \therefore\ 4(6x+1)\{2(1-x)\}=9x^2+18x+9;$
whence $57x^2-22x+1=0;$ *i.e.* $(3x-1)(19x-1)=0$, &c.

32. $4(2x+3)-4\sqrt{(2x+3)}\sqrt{(5x+1)}+5x+1=2(x-1);$
whence $-4\sqrt{(2x+3)}\sqrt{(5x+1)}=-11x-15;$
$16(2x+3)(5x+1)=121x^2+330x+225;$
whence $39x^2-58x-177=0;$ *i.e.* $(39x+59)(x-3)=0$, &c.

33. $\sqrt{(x-1)(x-2)}=\sqrt{2}-\sqrt{(x-3)(x-4)};$
$\therefore\ (x-1)(x-2)=2-2\sqrt{2}\sqrt{(x-3)(x-4)}+(x-3)(x-4);$
whence $2x-6=-\sqrt{2}\sqrt{(x-3)(x-4)};\ \therefore\ 4x^2-24x+36=2(x-3)(x-4);$
whence $2x^2-10x+12=0;$ *i.e.* $2(x-2)(x-3)=0$, &c.

34. $x+1+2\sqrt{(x+1)}\dfrac{1}{\sqrt{(x+1)}}+\dfrac{1}{x+1}=4;$
i.e. $x+1+2+\dfrac{1}{x+1}=4;\ \therefore\ x-1+\dfrac{1}{x+1}=0;$ whence $x=0.$

35. $\sqrt{(x+1)}-\dfrac{3}{\sqrt{(x+1)}}=-\sqrt{x};$
$\therefore\ (x+1)-2\sqrt{(x+1)}\dfrac{3}{\sqrt{(x+1)}}+\dfrac{9}{x+1}=x,$
i.e. $(x+1)-6+\dfrac{9}{x+1}=x;\ \therefore\ -5+\dfrac{9}{x+1}=0$, &c.

36. $\sqrt{(3+x)}=\dfrac{5}{\sqrt{x}}-\sqrt{x};\ \therefore\ 3+x=\dfrac{25}{x}-2\,\dfrac{5}{\sqrt{x}}\cdot\sqrt{x}+x;$
i.e. $3+x=\dfrac{25}{x}-10+x;\ \therefore\ 13-\dfrac{25}{x}=0;\ \therefore\ x=\dfrac{25}{13}.$

37. $6x+\dfrac{3}{2}=x+4+2\sqrt{(x+4)}\dfrac{3}{\sqrt{(x+4)}}+\dfrac{9}{x+4};$
i.e. $6x+\dfrac{3}{2}=x+4+6+\dfrac{9}{x+4};\ \therefore\ 5x-\dfrac{17}{2}=\dfrac{9}{x+4};$
$\therefore\ 10x(x+4)-17(x+4)=18;$
whence $10x^2+23x-86=0;$ *i.e.* $(10x+43)(x-2)=0$, &c.

38. $a-x+2\sqrt{(a-x)}\sqrt{(b-x)}+b-x=a+b-2x$;
whence $2\sqrt{(a-x)}\sqrt{(b-x)}=0$; $\therefore (a-x)(b-x)=0$, &c.

39. $\sqrt{(ax+b^2)}=a-b-\sqrt{(bx+a^2)}$;
$\therefore ax+b^2=a^2-2ab+b^2-2(a-b)\sqrt{(bx+a^2)}+bx+a^2$;
$\therefore x(a-b)-2a(a-b)=-2(a-b)\sqrt{(bx+a^2)}$;
$\therefore x-2a=-2\sqrt{(bx+a^2)}$; $\therefore x^2-4ax+4a^2=4(bx+a^2)$;
whence $x^2-4(a+b)x=0$; *i.e.* $x\{x-4(a+b)\}=0$, &c.

40. $a+x+2\sqrt{(a+x)}\sqrt{(b+x)}+b+x=a+b+2x$;
whence $2\sqrt{(a+x)}\sqrt{(b+x)}=0$; $\therefore (a+x)(b+x)=0$, &c.

41. $a-x+2\sqrt{(a-x)}\sqrt{(b-x)}+b-x=2a+2b$;
$\therefore 2\sqrt{(a-x)}\sqrt{(b-x)}=a+b+2x$;
$\therefore 4(a-x)(b-x)=a^2+2ab+b^2+4x(a+b)+4x^2$;
whence $8x(a+b)+(a^2-2ab+b^2)=0$, &c.

XLVII. Pages 174—176.

1. $(x-2)(x+2)=0$, *i.e.* $x^2-4=0$.

2. $(x-3)(x+4)=0$, *i.e.* $x^2+x-12=0$.

3. $(x+3)(x+2)=0$, *i.e.* $x^2+5x+6=0$.

4. $\left(x-\frac{1}{2}\right)\left(x+\frac{1}{2}\right)=0$, *i.e.* $x^2-\frac{1}{4}=0$, or $4x^2-1=0$.

5. $\left(x-\frac{1}{3}\right)\left(x+\frac{1}{4}\right)=0$, or $(3x-1)(4x+1)=0$, *i.e.* $12x^2-x-1=0$.

6. $\left(x+\frac{1}{3}\right)\left(x+\frac{1}{2}\right)=0$, or $(3x+1)(2x+1)=0$, *i.e.* $6x^2+5x+1=0$.

7. $x(x-3)=0$, *i.e.* $x^2-3x=0$. **8.** $x(x+4)=0$, *i.e.* $x^2+4x=0$.

9. $(x-5)(x+3)x=0$, *i.e.* $x^3-2x^2-15x=0$.

10. $(x-\sqrt{2})(x+\sqrt{2})=0$, *i.e.* $x^2-2=0$.

11. $(x-\sqrt{5})(x+\sqrt{5})=0$, *i.e.* $x^2-5=0$.

12. $x(x-\sqrt{3})(x+\sqrt{3})=0$, *i.e.* $x^3-3x=0$.

13. $(x-2+\sqrt{3})(x-2-\sqrt{3})=0$, *i.e.* $(x-2)^2-3=0$, &c.

14. $(x-5+\sqrt{7})(x-5-\sqrt{7})=0$, *i.e.* $(x-5)^2-7=0$, &c.

15. $(x-a+\sqrt{b})(x-a-\sqrt{b})=(x-a)^2-b=0$, &c.

18. From Art. 150, we have $\alpha+\beta=-\frac{b}{a}$ and $\alpha\beta=\frac{c}{a}$;

$$\therefore\ \frac{\alpha+\beta}{\alpha\beta}=-\frac{b}{a}\div\frac{c}{a}=-\frac{b}{c},\ i.e.\ \frac{1}{\beta}+\frac{1}{\alpha}=-\frac{b}{c}.$$

19. We have $$\alpha+\beta=-\frac{b}{a},\ \alpha\beta=\frac{c}{a}.$$

Hence (i) $\alpha^2+\beta^2=(\alpha+\beta)^2-2\alpha\beta=\frac{b^2}{a^2}-2\frac{c}{a}.$

(ii) $\frac{\alpha}{\beta}+\frac{\beta}{\alpha}=\frac{\alpha^2+\beta^2}{\alpha\beta}=\frac{(\alpha+\beta)^2-2\alpha\beta}{\alpha\beta}=\left(\frac{b^2}{a^2}-2\frac{c}{a}\right)\div\frac{c}{a}.$

(iii) $\frac{1}{\alpha^2}+\frac{1}{\beta^2}=\frac{\alpha^2+\beta^2}{\alpha^2\beta^2}=\frac{(\alpha+\beta)^2-2\alpha\beta}{\alpha^2\beta^2}=\left(\frac{b^2}{a^2}-2\frac{c}{a}\right)\div\frac{c^2}{a^2}.$

20. $\alpha+\beta=-4,\ \alpha\beta=2;\ \therefore\ \alpha^2+\beta^2=(\alpha+\beta)^2-2\alpha\beta=(-4)^2-4=12.$

21. $\alpha+\beta=-4p,\ \alpha\beta=p^2;\ \therefore\ \alpha^2+\beta^2=(\alpha+\beta)^2-2\alpha\beta=16p^2-2p^2=14p^2.$

22. If α, β be the roots of $x^2+ax+b=0$, $\alpha^2+\beta^2=(\alpha+\beta)^2-2\alpha\beta=a^2-2b$. If γ, δ be the roots of $x^2+3ax+b+4a^2=0$,

$$\gamma^2+\delta^2=(\gamma+\delta)^2-2\gamma\delta=(-3a)^2-2(b+4a^2)=a^2-2b.$$

23. $\alpha+\beta=-\frac{4}{3},\ \alpha\beta=\frac{a}{3}$; thus if the roots are equal, *i.e.* if $\beta=\alpha$, we have $$2\alpha=-\frac{4}{3},\ \text{whence } \alpha=-\frac{2}{3},\ \text{(i)};$$

and $$\alpha^2=\frac{a}{3},\ \text{whence and from (i) } \frac{4}{9}=\frac{a}{3};\ \therefore\ a=\frac{4}{3}.$$

Otherwise thus: condition for equal roots, viz. "$b^2-4ac=0$", (§ 145), becomes in this equation, $4^2-4(3a)=0$, whence $a=\frac{4}{3}$.

24. Condition for equal roots becomes in this equation, $(1+a)^2-16=0$; *i.e.* $\{(1+a)-4\}\{(1+a)+4\}=0$; whence $a=3$, or -5.

25. $\alpha+\beta=-\frac{60}{100}=-\frac{3}{5},\ \alpha\beta=\frac{a}{100}$; if one root is double the other, (say $\beta=2\alpha$), these equations become, $3\alpha=-\frac{3}{5}$,

and $$2\alpha^2=\frac{a}{100};\ \text{whence } 2\left(-\frac{1}{5}\right)^2=\frac{a}{100};\ \therefore\ a=8.$$

26. $\alpha+\beta=-p,\ \alpha\beta=q$; thus, if $9q=2p^2$,

$$9\alpha\beta=2(\alpha+\beta)^2,\ \text{whence } 2\alpha^2-5\alpha\beta+2\beta^2=0;$$

i.e. $(2\alpha-\beta)(\alpha-2\beta)=0$; whence either $2\alpha=\beta$, or $\alpha=2\beta$; *i.e.* one of the roots must be double of the other.

27. We have $\alpha+\beta=\frac{5}{2}$ and $\alpha\beta=\frac{3}{2}$. The equation whose roots are $\frac{\alpha}{\beta}$, $\frac{\beta}{\alpha}$, is $\left(x-\frac{\alpha}{\beta}\right)\left(x-\frac{\beta}{\alpha}\right)=0$; *i.e.* $x^2-\left(\frac{\alpha}{\beta}+\frac{\beta}{\alpha}\right)x+1=0$.

But $\frac{\alpha}{\beta}+\frac{\beta}{\alpha}=\frac{\alpha^2+\beta^2}{\alpha\beta}=\frac{(\alpha+\beta)^2-2\alpha\beta}{\alpha\beta}=\frac{(\alpha+\beta)^2}{\alpha\beta}-2=\frac{\left(\frac{5}{2}\right)^2}{\frac{3}{2}}-2=\frac{13}{6}$; thus required equation becomes $x^2-\frac{13}{6}x+1=0$; or $6x^2-13x+6=0$.

28. $\alpha+\beta=\frac{15}{2}$, $\alpha\beta=2$. Required equation is

$$\left(x-\frac{\alpha}{\beta}\right)\left(x-\frac{\beta}{\alpha}\right)=0;\ i.e.\ x^2-\left(\frac{\alpha}{\beta}+\frac{\beta}{\alpha}\right)x+1=0.$$

$$\frac{\alpha}{\beta}+\frac{\beta}{\alpha}=\frac{(\alpha+\beta)^2}{\alpha\beta}-2=\frac{\left(\frac{15}{2}\right)^2}{2}-2=\frac{209}{8}.$$

Whence required equation is, $x^2-\frac{209}{8}x+1=0$; or $8x^2-209x+8=0$.

29. $\alpha+\beta=11$, $\alpha\beta=22$; roots of required equation are $\alpha+\beta$, and $\frac{1}{\alpha}+\frac{1}{\beta}$, *i.e.* $\alpha+\beta$, and $\frac{\alpha+\beta}{\alpha\beta}$, *i.e.* 11, and $\frac{11}{22}$, *i.e.* 11 and $\frac{1}{2}$; thus required equation is $(x-11)\left(x-\frac{1}{2}\right)=0$; that is $x^2-\frac{23}{2}x+\frac{11}{2}=0$; or $2x^2-23x+11=0$.

30. $\alpha+\beta=-7$, $\alpha\beta=9$; required equation is

$$\left(x-\frac{\alpha+\beta}{\alpha}\right)\left(x-\frac{\alpha+\beta}{\beta}\right)=0,\ i.e.\ x^2-\left(\frac{\alpha+\beta}{\alpha}+\frac{\alpha+\beta}{\beta}\right)x+\frac{(\alpha+\beta)^2}{\alpha\beta}=0;$$

now
$$\frac{\alpha+\beta}{\alpha}+\frac{\alpha+\beta}{\beta}=\frac{(\alpha+\beta)^2}{\alpha\beta}=\frac{7^2}{9}=\frac{49}{9};$$

whence required equation becomes $x^2-\frac{49}{9}x+\frac{49}{9}=0$; or $9x^2-49x+49=0$.

31. If α, β are roots of $x^2+px+q=0$, $\alpha+\beta=-p$, $\alpha\beta=q$; also, equation whose roots are α^2, β^2, is $(x-\alpha^2)(x-\beta^2)=0$; or $x^2-(\alpha^2+\beta^2)x+\alpha^2\beta^2=0$; but $\alpha^2+\beta^2=(\alpha+\beta)^2-2\alpha\beta=p^2-2q$, and $\alpha^2\beta^2=q^2$; thus required equation becomes,

$$x^2-(p^2-2q)x+q^2=0.$$

32. We may write these two equations respectively as,

$$x^2+\frac{b^2}{a^2}x+\frac{c^2}{a^2}=0,\ \text{I., and}\ x^2+\frac{b}{a}x+\frac{c}{a}=0,\ \text{II.}$$

From 31, the equation whose roots are the squares of the roots of II. is $x^2-\left(\frac{b^2}{a^2}-2\frac{c}{a}\right)x+\frac{c^2}{a^2}=0$; this is identical with I., if

$$\frac{b^2}{a^2}=-\left(\frac{b^2}{a^2}-2\frac{c}{a}\right);\ i.e.\ \text{if } 2\frac{b^2}{a^2}-2\frac{c}{a}=0;\ \text{or, if } b^2-ac=0.$$

33. $\alpha+\beta=-\frac{q}{p}$, $\alpha\beta=\frac{r}{p}$; roots of required equation are

$$\alpha+\beta,\ \text{and } \frac{\alpha\beta}{\alpha+\beta},\ i.e.\ -\frac{q}{p},\ \text{and } \frac{\frac{r}{p}}{-\frac{q}{p}};\ i.e.\ -\frac{q}{p} \text{ and } -\frac{r}{q};$$

thus required equation is $\left(x+\frac{q}{p}\right)\left(x+\frac{r}{q}\right)=0$; or $x^2+\frac{q^2+pr}{pq}x+\frac{r}{p}$;
or multiplying by pq, $\quad pqx^2+(q^2+pr)x+qr=0$.

34. $\alpha+\beta=-\frac{b}{a}$, $\alpha\beta=\frac{c}{a}$; equation whose roots are $\alpha\beta$ and $\frac{1}{\alpha\beta}$,

$\left(i.e.\ \frac{c}{a},\ \text{and } \frac{a}{c}\right)$, is $\left(x-\frac{c}{a}\right)\left(x-\frac{a}{c}\right)=0$; or $x^2-\frac{a^2+c^2}{ac}x+1=0$;
or, multiplying by ac, $\quad acx^2-(a^2+c^2)x+ac=0$.

Also equation whose roots are $\alpha+\beta$, and $\frac{1}{\alpha+\beta}$,

$\left(i.e.\ -\frac{b}{a},\ \text{and } -\frac{a}{b}\right)$, is $\left(x+\frac{b}{a}\right)\left(x+\frac{a}{b}\right)=0$, or $x^2+\frac{a^2+b^2}{ab}x+1=0$;
or, multiplying by ab, $abx^2+(a^2+b^2)x+ab=0$.

35. Let α, β be roots of first equation; γ, δ of the second.
Then $\quad \alpha+\beta=-a$, $\alpha\beta=b$; $\gamma+\delta=-p$, $\gamma\delta=q$.
But $\alpha\sim\beta=\gamma\sim\delta$; $\therefore$ $(\alpha-\beta)^2=(\gamma-\delta)^2$; i.e. $(\alpha+\beta)^2-4\alpha\beta=(\gamma+\delta)^2-4\gamma\delta$;
i.e. $(-a)^2-4b=(-p)^2-4q$; i.e. $a^2-4b=p^2-4q$.

36. $(\alpha+\beta)^2=(-p)^2=p^2$; $(\alpha-\beta)^2=(\alpha+\beta)^2-4\alpha\beta=p^2-4q$; thus roots of required equation are p^2, and p^2-4q; thus equation is

$$(x-p^2)(x-\overline{p^2-4q})=0;\ i.e.\ x^2-(2p^2-4q)x+p^2(p^2-4q)=0.$$

37. Required equation is $\left\{x-\left(\alpha+\frac{1}{\beta}\right)\right\}\left\{x-\left(\beta+\frac{1}{\alpha}\right)\right\}=0$,

$$i.e.\ x^2-\left(\alpha+\frac{1}{\beta}+\beta+\frac{1}{\alpha}\right)x+\left(\alpha+\frac{1}{\beta}\right)\left(\beta+\frac{1}{\alpha}\right)=0;$$

now $$\alpha+\frac{1}{\beta}+\beta+\frac{1}{\alpha}=\alpha+\beta+\frac{\alpha+\beta}{\alpha\beta}=-p+\frac{-p}{q}=-\frac{p(q+1)}{q};$$

and $$\left(\alpha+\frac{1}{\beta}\right)\left(\beta+\frac{1}{\alpha}\right)=\frac{\alpha\beta+1}{\beta}\cdot\frac{\alpha\beta+1}{\alpha}=\frac{(\alpha\beta+1)^2}{\alpha\beta}=\frac{(q+1)^2}{q}.$$

Thus equation becomes,

$$x^2 - \left(-\frac{p(q+1)}{q} \right) x + \frac{(q+1)^2}{q} = 0 \text{; or, multiplying by } q,$$

$$qx^2 + p(q+1)x + (q+1)^2 = 0.$$

38. We have $\alpha + \beta = -\frac{b}{a}$, $\alpha\beta = \frac{c}{a}$.

The equation whose roots are

$$\frac{1}{\alpha+2\beta} \text{ and } \frac{1}{\beta+2\alpha} \text{ is } \left(x - \frac{1}{\alpha+2\beta} \right) \left(x - \frac{1}{\beta+2\alpha} \right) = 0,$$

or $$\{(\alpha+2\beta)x - 1\}\{(2\alpha+\beta)x - 1\} = 0,$$

that is $$(\alpha+2\beta)(2\alpha+\beta)x^2 - 3(\alpha+\beta)x + 1 = 0.$$

Now $(\alpha+2\beta)(2\alpha+\beta) = 2\alpha^2 + 2\beta^2 + 5\alpha\beta = 2(\alpha+\beta)^2 + \alpha\beta = 2\frac{b^2}{a^2} + \frac{c}{a}$.

Hence the equation required is

$$\left(2\frac{b^2}{a^2} + \frac{c}{a} \right) x^2 + 3\frac{b}{a} x + 1 = 0, \text{ or } (2b^2 + ac)x^2 + 3abx + a^2 = 0.$$

XLVIII. PAGES 181—182.

1. $(x^2-1)(x^2-4)=0$; *i.e.* $(x-1)(x+1)(x-2)(x+2)=0$, &c.

2. $(x^2-4)(x^2+2)=0$; *i.e.* $(x-2)(x+2)(x-\sqrt{-2})(x+\sqrt{-2})$†$=0$, &c.

3. $(x^2-1)(x^2-9)=0$; *i.e.* $(x-1)(x+1)(x-3)(x+3)=0$, &c.

4. $(x^2-9)(x^2+2)=0$; *i.e.* $(x-3)(x+3)(x-\sqrt{-2})(x+\sqrt{-2})=0$, &c.

5. Multiply by $9x^2$; thus $x^2(x^2-1)+9-9x^2=0$;

or $$x^4 - 10x^2 + 9 = 0;\ i.e.\ (x^2-1)(x^2-9)=0, \text{ \&c.}$$

6. Clear of fractions; $x^4 + 100 = 29x^2$;

$$\therefore x^4 - 29x^2 + 100 = 0;\ i.e.\ (x^2-4)(x^2-25)=0;$$

$$\therefore (x-2)(x+2)(x-5)(x+5)=0, \text{ \&c.}$$

7. $x^4 + 1 = \left(a^2 + \frac{1}{a^2}\right)x^2$; $\therefore x^4 - \left(a^2 + \frac{1}{a^2}\right)x^2 + 1 = 0$;

i.e. $(x^2-a^2)\left(x^2 - \frac{1}{a^2}\right)=0$; *i.e.* $(x-a)(x+a)\left(x-\frac{1}{a}\right)\left(x+\frac{1}{a}\right)=0$, &c.

8. $x^4 - 1 = \left(a^2 - \frac{1}{a^2}\right)x^2$; $\therefore x^4 - \left(a^2 - \frac{1}{a^2}\right)x^2 - 1 = 0$;

i.e. $(x^2-a^2)\left(x^2 + \frac{1}{a^2}\right)=0$; *i.e.* $(x-a)(x+a)\left(x - \frac{\sqrt{-1}}{a}\right)\left(x + \frac{\sqrt{-1}}{a}\right)=0$, &c.

† $x^2 + 2 = x^2 - (-2) = (x - \sqrt{-2})(x + \sqrt{-2})$.

9. $x^8+1=\left(a^4+\frac{1}{a^4}\right)x^4;\ \therefore\ x^8-\left(a^4+\frac{1}{a^4}\right)x^4+1=0;$

i.e. $(x^4-a^4)\left(x^4-\frac{1}{a^4}\right)=0;$ *i.e.* $(x^2-a^2)(x^2+a^2)\left(x^2-\frac{1}{a^2}\right)\left(x^2+\frac{1}{a^2}\right)=0;$

i.e. $(x-a)(x+a)(x-a\sqrt{-1})(x+a\sqrt{-1})\left(x-\frac{1}{a}\right)\left(x+\frac{1}{a}\right)$

$$\left(x-\frac{\sqrt{-1}}{a}\right)\left(x+\frac{\sqrt{-1}}{a}\right)=0, \text{ \&c.}$$

10. Let $x^2-x=y$; then $y^2-8y+12=0$;

i.e. $(y-2)(y-6)=0;\ \therefore\ y=2$, or 6.

Thus, either $x^2-x=2$, whence $x^2-x-2=0$, *i.e.* $(x+1)(x-2)=0$,

whence $x=-1$, or 2;

or $x^2-x=6$, whence $x^2-x-6=0$; *i.e.* $(x-3)(x+2)=0$,

whence $x=3$, or -2.

11. Let $x^2+x=y$; then $y^2-22y+40=0$;

i.e. $(y-2)(y-20)=0;\ \therefore\ y=2$, or 20.

Thus, either $x^2+x=2$, whence $x^2+x-2=0$, *i.e.* $(x+2)(x-1)=0$, &c.,

or $x^2+x=20$; whence $x^2+x-20=0$, *i.e.* $(x+5)(x-4)=0$, &c.

12. Let $x^2+x=y$; then $y(y+1)=42;\ \therefore\ y^2+y-42=0;$

i.e. $(y-6)(y+7)=0;\ \therefore\ y=6$, or -7.

Thus either $x^2+x=6$, whence $x^2+x-6=0$, *i.e.* $(x+3)(x-2)=0$, &c.,

or $x^2+x=-7$, whence $x^2+x+7=0$, *i.e.* $x^2+x+\left(\frac{1}{2}\right)^2-\left\{\left(\frac{1}{2}\right)^2-7\right\}=0;$

$$i.e.\ \left(x+\frac{1}{2}\right)^2-\left(-\frac{27}{4}\right)=0;$$

$$i.e.\ \left(x+\frac{1}{2}-\sqrt{\frac{-27}{4}}\right)\left(x+\frac{1}{2}+\sqrt{\frac{-27}{4}}\right)=0, \text{ \&c.}$$

13. Put $\frac{x^2}{x+1}=y$; then $y+\frac{1}{y}=2$; whence $y^2-2y+1=0;\ \therefore\ y=1$.

Thus $\frac{x^2}{x+1}=1$; whence $x^2-x-1=0$;

$$i.e.\ \left(x-\frac{1}{2}\right)^2-\left(\frac{\sqrt{5}}{2}\right)^2=0;\ \text{whence } x=\frac{1}{2}\pm\frac{\sqrt{5}}{2}.$$

14. Put $\frac{x}{x^2+1}=y$; then $y+\frac{1}{y}=\frac{5}{2}$;

whence $2y^2-5y+2=0$; *i.e.* $(2y-1)(y-2)=0;\ \therefore\ y=\frac{1}{2}$, or 2.

Thus, either $\dfrac{x}{x^2+1}=\dfrac{1}{2}$, whence $x^2-2x+1=0$; *i.e.* $(x-1)^2=0$, &c.

or $$\frac{x}{x^2+1}=2, \text{ whence } x^2-\frac{1}{2}x+1=0;$$

$$i.e.\ \left(x-\frac{1}{4}\right)^2-\frac{-15}{16}=0; \text{ whence } x=\frac{1}{4}\pm\frac{\sqrt{-15}}{4}.$$

15. Put $\dfrac{x^2}{x-1}=y$; then $y+\dfrac{1}{y}=\dfrac{17}{4}$; whence $4y^2-17y+4=0$;

$$i.e.\ (4y-1)(y-4)=0; \quad \therefore\ y=\frac{1}{4}, \text{ or } 4.$$

Thus either $$\frac{x^2}{x-1}=\frac{1}{4}, \text{ whence } x^2-\frac{1}{4}x+\frac{1}{4}=0;$$

$$i.e.\ \left(x-\frac{1}{8}\right)^2-\left(\frac{-15}{64}\right)=0; \text{ whence } x=\frac{1}{8}\pm\frac{\sqrt{-15}}{8};$$

or $$\frac{x^2}{x-1}=4, \text{ whence } x^2-4x+4=0,\ i.e.\ (x-2)^2=0, \text{ \&c.}$$

16. Put $x^2+x=y$; then $(y+1)\left(y-\dfrac{3}{2}\right)+1=0$; whence $y^2-\dfrac{1}{2}y-\dfrac{1}{2}=0$;

$$i.e.\ (y-1)\left(y+\frac{1}{2}\right)=0; \quad \therefore\ y=1, \text{ or } -\frac{1}{2}.$$

Thus, either $x^2+x=1$, whence $\left(x+\dfrac{1}{2}\right)^2-\dfrac{5}{4}=0$, $\therefore\ x=-\dfrac{1}{2}\pm\dfrac{\sqrt{5}}{2}$;

or $x^2+x=-\dfrac{1}{2}$, whence $\left(x+\dfrac{1}{2}\right)^2-\left(-\dfrac{1}{4}\right)=0$, $\therefore\ x=-\dfrac{1}{2}\pm\dfrac{\sqrt{-1}}{2}$.

17. Put $x^2+x=y$; then $(y+1)(y+2)=12$; $\therefore\ y^2+3y-10=0$;

whence $$y=2, \text{ or } -5.$$

Thus, either $x^2+x=2$, whence $x^2+x-2=0$, *i.e.* $(x-1)(x+2)=0$, &c.

or $x^2+x=-5$, whence $\left(x+\dfrac{1}{2}\right)^2-\left(\dfrac{-19}{4}\right)=0$; $\therefore\ x=-\dfrac{1}{2}\pm\dfrac{\sqrt{-19}}{2}$.

18. Put $x+\dfrac{1}{x}=y$; then $y^2+4y=12$; whence $y=2$, or -6.

Thus, either $x+\dfrac{1}{x}=2$; whence $x^2-2x+1=0$; $\therefore\ x=1$;

or $x+\dfrac{1}{x}=-6$; whence $x^2+6x+1=0$; *i.e.* $(x+3)^2-8=0$, &c.

19. Let $\dfrac{x^2+2}{x^2+4x+1}=y$; then $y+\dfrac{1}{y}=\dfrac{5}{2}$; whence $2y^2-5y+2=0$;

whence $$y=2 \text{ or } \frac{1}{2}.$$

Thus, either $\dfrac{x^2+2}{x^2+4x+1}=2$, whence $x^2+8x=0$, *i.e.* $x(x+8)=0$, &c.;

or $\dfrac{x^2+2}{x^2+4x+1}=\dfrac{1}{2}$, whence $x^2-4x+3=0$; *i.e.* $(x-1)(x-3)=0$, &c.

20. Let $x^2+x=y$; then $y+1=\dfrac{42}{y}$; $\therefore\ y^2+y-42=0$; whence proceed as in 12.

21. Let $x^2-2x-3=y^2$; then $x^2-2x=y^2+3$; thus $2(y^2+3)-y=9$;

whence $$2y^2-y-3=0;\quad \therefore\ y=-1, \text{ or } \frac{3}{2}.$$

Thus, either $x^2-2x-3=(-1)^2$, whence $x^2-2x-4=0$;

i.e. $(x-1)^2-5=0$, &c.,

or $x^2-2x-3=\left(\dfrac{3}{2}\right)^2$, whence $x^2-2x-\dfrac{21}{4}=0$; *i.e.* $(x-1)^2-\dfrac{25}{4}=0$, &c.

22. Let $x^2+3x+7=y^2$; then we have $x^2+3x=y^2-7$;

but $$x^2+3x-23+\sqrt{x^2+3x+7}=0;\quad \therefore\ y^2-7-23+y=0;$$

i.e. $y^2+y-30=0$, whence $y=5$, or -6.

Thus, either $x^2+3x+7=5^2$, whence $x^2+3x-18=0$; $\therefore\ x=3$, or -6;

or $x^2+3x+7=(-6)^2$, whence $x^2+3x-29=0$; $\therefore\ \left(x+\dfrac{3}{2}\right)^2-\dfrac{125}{4}=0$, &c.

23. Let $x^2+3x+1=y^2$; then $x^2+3x=y^2-1$; thus $2(y^2-1)=1-y$;

whence $$2y^2+y-3=0;\quad \therefore\ y=1, \text{ or } -\frac{3}{2}.$$

Thus, either $x^2+3x+1=1^2$; whence $x(x+3)=0$, &c.

or $x^2+3x+1=\left(-\dfrac{3}{2}\right)^2$; whence $\left(x+\dfrac{3}{2}\right)^2-\dfrac{14}{4}=0$, &c.

24. Let $x^2+6x=y^2$; then we have $x^2+6x-24+\sqrt{(4x^2+24x)}=0$; whence $y^2-24+\sqrt{(4y^2)}=0$; *i.e.* $y^2+2y-24=0$; whence $y=4$, or -6.

Thus either $x^2+6x=4^2$, whence $x=2$, or -8;

or $x^2+6x=(-6)^2$; whence $(x+3)^2-45=0$, &c.

25. Let $2x^2-11x+15=y^2$; then we have $2x^2-11x=y^2-15$.

But we have $4x^2-22x+24-\sqrt{(2x^2-11x+15)}=60$;

$\therefore\ 2(y^2-15)+24-y=60$; whence $2y^2-y-66=0$; $\therefore\ y=6$, or $-\dfrac{11}{2}$.

Thus, either $2x^2-11x+15=6^2$, whence $2x^2-11x-21=0$,

$$i.e.\ (2x+3)(x-7)=0,\ \&c.,$$

or $$2x^2-11x+15=\left(-\frac{11}{2}\right)^2,\ \text{whence}\ \left(x-\frac{11}{4}\right)^2-\frac{243}{16}=0,\ \&c.$$

26. Let $2x^2-5x+6=y^2$; then we have $2x^2-5x+6+\sqrt{2x^2-5x+6}=6$;

$$\therefore\ y^2+y=6;\ \text{whence}\ y=2,\ \text{or}\ -3.$$

Thus, either $2x^2-5x+6=4$; whence $(2x-1)(x-2)=0$, &c.

or $$2x^2-5x+6=9,\ \text{whence}\ (2x+1)(x-3)=0,\ \&c.$$

27. $x^2+3x-10-36=\sqrt{x^2+3x-4}$; let $x^2+3x-4=y^2$;

then $$y^2-42=y;\ \text{whence}\ y=7,\ \text{or}\ -6.$$

Thus, either $x^2+3x-4=49$; whence $\left(x+\frac{3}{2}\right)^2-\frac{221}{4}=0$;

whence $$x=-\frac{3}{2}\pm\frac{1}{2}\sqrt{221};$$

or $$x^2+3x-4=36;\ \text{whence}\ (x+8)(x-5)=0,\ \&c.$$

28. Since $x=1$ is a root of the equation, x^3-2x+1 must be divisible by $x-1$; dividing, we obtain, $x^3-2x+1=(x-1)(x^2+x-1)$; thus the equation may be written $(x-1)(x^2+x-1)=0$; *i.e.* $(x-1)\left\{\left(x+\frac{1}{2}\right)^2-\frac{5}{4}\right\}=0$;

$$i.e.\ (x-1)\left\{\left(x+\frac{1}{2}\right)-\frac{\sqrt{5}}{2}\right\}\left\{\left(x+\frac{1}{2}\right)+\frac{\sqrt{5}}{2}\right\}=0,\ \&c.$$

29. $(x-1)$ must be a factor, and, by division, we find that the equation is equivalent to $(x-1)(x^2+x-4)=0$; *i.e.* $(x-1)\left\{\left(x+\frac{1}{2}\right)^2-\frac{17}{4}\right\}=0$;

$$i.e.\ (x-1)\left\{\left(x+\frac{1}{2}\right)-\frac{\sqrt{17}}{2}\right\}\left\{\left(x+\frac{1}{2}\right)+\frac{\sqrt{17}}{2}\right\}=0,\ \&c.$$

30. $x-3$ is a factor; whence by division, equation becomes

$$(x-3)(x^2+3x-40)=0;\ i.e.\ (x-3)(x+8)(x-5)=0,\ \&c.$$

31. $x-3$ is a factor; and we have $(x-3)(x^2-7)=0$;

$$i.e.\ (x-3)(x-\sqrt{7})(x+\sqrt{7})=0,\ \&c.$$

32. $x+1$ is a factor; and we have $(x+1)(x^2-3x-4)=0$;

$$i.e.\ (x+1)(x+1)(x-4)=0,\ \&c.$$

33. $x-2$ is a factor; and we have

$$(x-2)(5x^2-5x-7)=0;\ \therefore\ (x-2)\left(x^2-x-\frac{7}{5}\right)=0;$$

$$i.e.\ (x-2)\left\{\left(x-\frac{1}{2}\right)^2-\frac{33}{20}\right\}=0;\ i.e.\ (x-2)\left\{\left(x-\frac{1}{2}\right)^2-\frac{165}{100}\right\}=0,$$

[we multiplied numerator and denominator of $\frac{33}{20}$ by 5 in order to reduce it to a fraction whose denominator was a perfect square].

Thus $$(x-2)\left\{\left(x-\frac{1}{2}\right)-\frac{\sqrt{165}}{10}\right\}\left\{\left(x-\frac{1}{2}\right)+\frac{\sqrt{165}}{10}\right\}=0, \text{ \&c.}$$

34.
$$\begin{array}{l}
x^3-4x+15\ )\ x^3+2x^2+9\ (\ 1\\
\qquad\qquad\quad\ \ \underline{x^3-4x+15}\\
\qquad\qquad\quad\ \ 2\ )\ \underline{2x^2+4x-6}\\
\qquad\qquad\qquad\quad x^2+2x-3\ )\ x^3-4x+15\ (\ x-2\\
\qquad\qquad\qquad\qquad\qquad\qquad\ \ \underline{x^3+2x^2-3x}\\
\qquad\qquad\qquad\qquad\qquad\qquad\quad -2x^2-x+15\\
\qquad\qquad\qquad\qquad\qquad\qquad\quad \underline{-2x^2-4x+6}\\
\qquad\qquad\qquad\qquad\qquad\qquad\qquad\ 3\ )\ \underline{3x+9}\\
\qquad\qquad\qquad\qquad\qquad\qquad\qquad\quad x+3\ )\ x^2+2x-3\ (\ x-1\\
\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\ \ \underline{x^2+3x}\\
\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\quad -x-3\\
\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\quad -x-3
\end{array}$$

Thus H.C.F. of the two expressions is $(x+3)$; they will $\therefore$ both vanish when this is zero, *i.e.* when $x=-3$.

35.
$$\begin{array}{l}
2x^3-x^2-76x-105\ )\ 2x^3+21x^2+34x-105\ (\ 1\\
\qquad\qquad\qquad\qquad\ \ \underline{2x^3-x^2-76x-105}\\
\qquad\qquad\qquad\qquad\ 22x\ )\ \underline{22x^2+110x}\\
\qquad\qquad\qquad\qquad\qquad\quad x+5\ )\ 2x^3-x^2-76x-105\ (\ 2x^2-11x-21\\
\qquad\qquad\qquad\qquad\qquad\qquad\qquad\ \underline{2x^3+10x^2}\\
\qquad\qquad\qquad\qquad\qquad\qquad\qquad\quad -11x^2-76x-105\\
\qquad\qquad\qquad\qquad\qquad\qquad\qquad\quad \underline{-11x^2-55x}\\
\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad -21x-105\\
\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad -21x-105
\end{array}$$

Thus H.C.F. is $x+5$; $\therefore$ both equations are satisfied when $x=-5$.

36. (i) By trial we find equation is satisfied by $x=1$; $\therefore$ $x-1$ is a factor; and, by division, equation becomes

$$(x-1)(x^2+x-20)=0;\ \textit{i.e.}\ (x-1)(x-4)(x+5)=0, \text{ \&c.}$$

(ii) $x^2(x+2)-9(x+2)=0$; *i.e.* $(x+2)(x^2-9)=0$;
i.e. $(x+2)(x-3)(x+3)=0$, &c.

(iii) By trial we find equation is satisfied by $x=2$; $\therefore$ $x-2$ is a factor; and, by division, equation becomes

$$(x-2)(x^2+x-20)=0;\ \textit{i.e.}\ (x-2)(x-4)(x+5)=0, \text{ \&c.}$$

(iv) By trial we find equation is satisfied by $x=-2$, thus $(x+2)$ is a factor; and, by division, we obtain

$$(x+2)(x^2-5x-50)=0;\ \textit{i.e.}\ (x+2)(x-10)(x+5)=0, \text{ \&c.}$$

(v) By trial we find equation is satisfied by $x=2$; thus $(x-2)$ is a factor; and, by division, we obtain

$$(x-2)(x^2+2x-15)=0;\ \textit{i.e.}\ (x-2)(x+5)(x-3)=0, \text{ \&c.}$$

(vi) By trial, $x=1$ satisfies equation; thus $(x-1)$ is a factor; whence, by division,

$$(x-1)(2x^2+9x+4)=0; \ i.e. \ (x-1)(2x+1)(x+4)=0, \text{ \&c.}$$

(vii) $4x^2(x+1)-9(x+1)=0$; *i.e.* $(x+1)(4x^2-9)=0$;

$$i.e. \ (x+1)(2x-3)(2x+3)=0, \text{ \&c.}$$

XLIX. PAGE 185.

1. From I, $x=6-y$; substituting in II, $(6-y)^2-y^2=24$;

i.e. $36-12y=24$, whence $y=1$. Then $x=6-y=5$.

2. From I, $x=y+10$; substituting in II, $(y+10)^2+y^2=58$;

i.e. $2y^2+20y+42=0$; whence $y=-3$, or -7.

If $y=-3$, $x=y+10=7$. If $y=-7$, $x=y+10=3$.

3. From I, $3x=10-3y$, $\therefore x=\frac{10-3y}{3}$; thus from II, $\left(\frac{10-3y}{3}\right)y=1$;

whence $-3y^2+10y-3=0$; *i.e.* $-(3y-1)(y-3)=0$; $\therefore y=\frac{1}{3}$ or 3.

From II, $x=\frac{1}{y}$; thus if $y=\frac{1}{3}$, $x=3$; if $y=3$, $x=\frac{1}{3}$.

4. From I, $x=-\frac{3y}{2}$; thus from II,

$$4\left(-\frac{3y}{2}\right)^2+9\left(-\frac{3y}{2}\right)y+9y^2=72; \ i.e. \ 9y^2-\frac{27}{2}y^2+9y^2=72,$$

$$\therefore \frac{9}{2}y^2=72, \text{ whence } y=\pm 4.$$

If $y=4$, $x=-\frac{3y}{2}=-6$. If $y=-4$, $x=-\frac{3y}{2}=6$.

5. From I, $x=\frac{5}{2}y$; thus from II, $\left(\frac{5}{2}y\right)^2-3y^2=13$;

$$i.e. \ \frac{13}{4}y^2=13, \text{ whence } y=\pm 2.$$

If $y=2$, $x=\frac{5}{2}y=5$. If $y=-2$, $x=\frac{5}{2}y=-5$.

6. From I, $x=15-y$; thus from II, $4(15-y)-4y=(15-y)y$; whence $y^2-23y+60=0$; whence $y=3$ or 20.

If $y=3$, $x=15-y=12$. If $y=20$, $x=15-y=-5$.

7. From I, $x=\frac{y+5}{2}$; thus from II, $\frac{y+5}{2}+3y=2\left(\frac{y+5}{2}\right)y$;

whence $y^2+\frac{3}{2}y-\frac{5}{2}=0$; whence $y=1$ or $-\frac{5}{2}$.

If $y=1$, $x=\frac{y+5}{2}=3$. If $y=-\frac{5}{2}$, $x=\frac{y+5}{2}=\frac{5}{4}$.

8. From I, $x=\frac{5y+31}{3}$; thus from II,

$$\left(\frac{5y+31}{3}\right)^2+5\left(\frac{5y+31}{3}\right)y+25=y^2;$$

whence $91y^2+775y+1186=0$; *i.e.* $(91y+593)(y+2)=0$;

$$\therefore y=-2 \text{ or } -\frac{593}{91}.$$

If $y=-2$, $x=\frac{5y+31}{3}=7$. If $y=-\frac{593}{91}$, $x=\frac{5y+31}{3}=-\frac{48}{91}$.

9. From I, $x=\frac{5-2y}{3}$; thus from II,

$$\left(\frac{5-2y}{3}\right)^2-4\left(\frac{5-2y}{3}\right)y+5y^2=2;$$

whence $73y^2-80y+7=0$; *i.e.* $(73y-7)(y-1)=0$; $\therefore y=\frac{7}{73}$, or 1.

If $y=\frac{7}{73}$, $x=\frac{5-2y}{3}=\frac{117}{73}$. If $y=1$, $x=\frac{5-2y}{3}=1$.

10. From I, $x=\frac{25+7y}{2}$; thus from II,

$$5\left(\frac{25+7y}{2}\right)^2+4\left(\frac{25+7y}{2}\right)y+3y^2=23;$$

whence $313y^2+1950y+3033=0$; *i.e.* $(313y+1011)(y+3)=0$;

$$\therefore y=-3, \text{ or } -\frac{1011}{313}.$$

If $y=-3$, $x=\frac{25+7y}{2}=2$. If $y=-\frac{1011}{313}$, $x=\frac{25+7y}{2}=\frac{374}{313}$.

11. From I, $x=2-y$.
From II, $2y+3x=6xy$; $\therefore 2y+3(2-y)=6(2-y)y$;

whence $6y^2-13y+6=0$; *i.e.* $(3y-2)(2y-3)=0$; $\therefore y=\frac{2}{3}$, or $\frac{3}{2}$.

If $y=\frac{2}{3}$, $x=2-y=\frac{4}{3}$. If $y=\frac{3}{2}$, $x=2-y=\frac{1}{2}$.

12. From I, $x=7-2y$.

From II, $3y+6x=5xy$; $\therefore\ 3y+6(7-2y)=5(7-2y)y$;

whence $10y^2-44y+42=0$; *i.e.* $2(y-3)(5y-7)=0$; $\therefore\ y=3$, or $\frac{7}{5}$.

If $y=3$, $x=7-2y=1$. If $y=\frac{7}{5}$, $x=7-2y=\frac{21}{5}$.

13. From I, $x=5-y$.

From II, $6y+6x=5xy$; $\therefore\ 6y+6(5-y)=5(5-y)y$;

whence $5y^2-25y+30=0$; *i.e.* $5(y-2)(y-3)=0$; $\therefore\ y=2$, or 3.

If $y=2$, $x=5-y=3$. If $y=3$, $x=5-y=2$.

14. From I, $x=1+y$.

From II, $6x^2-6y^2=5xy$; $\therefore\ 6(1+y)^2-6y^2=5(1+y)y$;

whence $-5y^2+7y+6=0$; *i.e.* $-(5y+3)(y-2)=0$; $\therefore\ y=-\frac{3}{5}$, or 2.

If $y=-\frac{3}{5}$, $x=1+y=\frac{2}{5}$. If $y=2$, $x=1+y=3$.

15. From I, $x=\frac{y+4}{2}$.

From II, $3y+4x=xy$; $\therefore\ 3y+4\left(\frac{y+4}{2}\right)=\frac{y+4}{2}y$;

$\therefore\ 6y+4(y+4)=(y+4)y$; whence $-y^2+6y+16=0$; whence $y=8$, or -2.

If $y=8$, $x=\frac{y+4}{2}=6$. If $y=-2$, $x=\frac{y+4}{2}=1$.

16. From I, $x=\frac{3y-7}{7}$.

From II, $5y-14x=6xy$; $\therefore\ 5y-14\left(\frac{3y-7}{7}\right)=6\left(\frac{3y-7}{7}\right)y$;

$\therefore\ 35y-14(3y-7)=6(3y-7)y$; whence $-18y^2+35y+98=0$;

i.e. $-(2y-7)(9y+14)=0$; thus $y=\frac{7}{2}$ or $-\frac{14}{9}$.

If $y=\frac{7}{2}$, $x=\frac{3y-7}{7}=\frac{1}{2}$. If $y=-\frac{14}{9}$, $x=\frac{3y-7}{7}=-\frac{5}{3}$.

17. (I − II); By subtraction, $2x=7$; $\therefore\ x=\frac{7}{2}$.

By addition, $2xy=23$; $\therefore\ y=\frac{23}{2x}=\frac{23}{7}$.

18. Multiply the first equation by 2; then by subtraction

$$4x+y=7; \quad \therefore\ y=7-4x.$$

Then, from I, $x(7-4x)+2x=5$; $\therefore\ -4x^2+9x-5=0$;

$$i.e.\ -(4x-5)(x-1)=0; \quad \therefore\ x=\frac{5}{4}, \text{ or } 1.$$

If $x=\frac{5}{4}$, $y=7-4x=2$. If $x=1$, $y=7-4x=3$.

19. By addition, $2x^2+x=78$; $\therefore\ 2x^2+x-78=0$;

$$i.e.\ (2x+13)(x-6)=0; \quad \therefore\ x=-\frac{13}{2}, \text{ or } 6.$$

From I, if $x=-\frac{13}{2}$, $\frac{169}{4}-y=29$; $\therefore\ y=\frac{53}{4}$;

if $$x=6,\ 36-y=29; \quad \therefore\ y=7.$$

20. $\text{II}\times 2+\text{I}\times 3$, gives $7x^2-x+4=12$; $\therefore\ 7x^2-x-8=0$;

$$i.e.\ (7x-8)(x+1)=0; \quad \therefore\ x=\frac{8}{7}, \text{ or } -1.$$

From I, if $x=\frac{8}{7}$, $\frac{64}{49}+\frac{24}{7}-2y=4$, whence $y=\frac{18}{49}$;

if $$x=-1,\ 1-3-2y=4; \text{ whence } y=-3.$$

L. PAGES 188, 189.

1. From I, $x(x-2y)=0$; $\therefore\ x=0$, or $x=2y$.
Thus from II, (i) if $x=0$, $9y^2=225$; $\therefore\ 9y^2-225=0$,

$$i.e.\ 9(y-5)(y+5)=0, \&c.$$

$$\text{(ii) if } x=2y,\ 16y^2+9y^2=225; \quad \therefore\ 25y^2-225=0;$$

$$i.e.\ 25(y-3)(y+3)=0; \quad \therefore\ y=\pm 3; \quad \therefore\ x=2y=\pm 6.$$

2. From I, $x(2x-3y)=0$; $\therefore\ x=0$, or $x=\frac{3y}{2}$.

Thus from II, (i) if $x=0$, $y^2=34$, whence $y=\pm\sqrt{34}$.

$$\text{(ii) if } x=\frac{3y}{2},\ y^2+\frac{15y^2}{2}=34; \quad \therefore\ 2y^2+15y^2=68;$$

whence $$y=\pm 2; \quad \therefore\ x=\frac{3y}{2}=\pm 3.$$

3. From I and II, $\dfrac{x^2+3xy}{y^2-xy}=\dfrac{45}{4}$;

whence $$4(x^2+3xy)=45(y^2-xy); \quad \therefore\ 4x^2+57xy-45y^2=0;$$

$$i.e.\ (4x-3y)(x+15y)=0; \quad \therefore\ x=\frac{3y}{4}, \text{ or } x=-15y.$$

Thus from II, (i) if $x=\frac{3y}{4}$, $y^2-\frac{3y^2}{4}=4$, whence $y=\pm 4$,

and $x=\frac{3y}{4}=\pm 3$;

or (ii), if $x=-15y$, $y^2+15y^2=4$, whence $y=\pm\frac{1}{2}$,

and $x=-15y=\mp\frac{15}{2}$.

4. From I and II, $22(x^2-xy)=63(y^2+xy)$;
$\therefore$ $22x^2-85xy-63y^2=0$; *i.e.* $(2x-9y)(11x+7y)=0$; $\therefore$ $x=\frac{9y}{2}$, or $x=-\frac{7y}{11}$.

Thus from II, (i), if $x=\frac{9y}{2}$, $y^2+\frac{9y^2}{2}=22$, whence $y=\pm 2$,

and $x=\frac{9y}{2}=\pm 9$;

or (ii), if $x=-\frac{7y}{11}$, $y^2-\frac{7y^2}{11}=22$, whence $y=\pm\frac{11}{\sqrt{2}}$,

and $x=-\frac{7y}{11}=\mp\frac{7}{\sqrt{2}}$.

5. $4(x^2+xy)=3(2y^2+3xy)$;
thus $4x^2-5xy-6y^2=0$; *i.e.* $(4x+3y)(x-2y)=0$; $\therefore$ $x=-\frac{3y}{4}$, or $x=2y$.

Then, from II, (i) if $x=-\frac{3y}{4}$, $2y^2-\frac{9y^2}{4}=32$,

whence $y=\pm 8\sqrt{-2}$; $\therefore$ $x=-\frac{3y}{4}=\mp 6\sqrt{-2}$;

(ii) if $x=2y$, $2y^2+6y^2=32$, whence $y=\pm 2$;
$\therefore$ $x=2y=\pm 4$.

6. $(x^2-3xy)+10(4y^2-xy)=0$;
thus $-x^2+13xy-40y^2=0$; *i.e.* $-(x-5y)(x-8y)=0$; $\therefore$ $x=5y$, or $x=8y$.
Thus from II, (i) if $x=5y$, $4y^2-5y^2=-1$, whence $y=\pm 1$;
$\therefore$ $x=5y=\pm 5$;

(ii) if $x=8y$, $4y^2-8y^2=-1$; whence $y=\pm\frac{1}{2}$;

$\therefore$ $x=8y=\pm 4$.

7. $20(x^2+xy-2y^2)=-11(xy+3y^2)$;
$\therefore$ $20x^2+31xy-7y^2=0$; *i.e.* $(4x+7y)(5x-y)=0$; $\therefore$ $x=-\frac{7y}{4}$, or $x=\frac{y}{5}$.

Thus from II, (i) if $x=-\frac{7y}{4}$, $-\frac{7y^2}{4}+3y^2=80$, whence $y=\pm 8$;

$$\therefore\ x=-\frac{7y}{4}=\mp 14;$$

(ii) if $x=\frac{y}{5}$, $\frac{y^2}{5}+3y^2=80$; whence $y=\pm 5$;

$$\therefore\ x=\frac{y}{5}=\pm 1.$$

8. $6(x^2+3xy)=7(y^2+xy)$;

$\therefore\ 6x^2+11xy-7y^2=0$; *i.e.* $(2x-y)(3x+7y)=0$, $\therefore\ x=\frac{y}{2}$, or $x=-\frac{7y}{3}$.

Thus from II, (i) if $x=\frac{y}{2}$, $y^2+\frac{y^2}{2}=6$, whence $y=\pm 2$; $\therefore\ x=\frac{y}{2}=\pm 1$;

(ii) if $x=-\frac{7y}{3}$, $y^2-\frac{7y^2}{3}=6$, whence $y=\pm 3\sqrt{-\frac{1}{2}}$;

$$\therefore\ x=-\frac{7y}{3}=\mp 7\sqrt{-\frac{1}{2}}.$$

9. $(\text{II}\times 3-\text{I})$; $8x^2=200$; whence $x=\pm 5$; thus from II, $75-y^2=71$; whence $y=\pm 2$.

N.B. Here either sign may go with either; *e.g.* $x=-5$, $y=+2$, is a solution.

10. $(\text{I}\times 5-\text{II}\times 3)$; $11x^2=44$; $\therefore\ x=\pm 2$;
from I, (i) if $x=\ \ 2$, $\ 6y+4=10$, $\therefore\ y=1$;
(ii) if $x=-2$, $-6y+4=10$, $\therefore\ y=-1$.

11. $115(x^2+3xy)=54(xy+4y^2)$;

$\therefore\ 115x^2+291xy-216y^2=0$; *i.e.* $(5x-3y)(23x+72y)=0$;

$$\therefore\ x=\frac{3y}{5},\ \text{or}\ x=-\frac{72y}{23}.$$

Thus from II, (i) if $x=\frac{3y}{5}$, $\frac{3y^2}{5}+4y^2=115$, whence $y=\pm 5$;

$$\therefore\ x=\frac{3y}{5}=\pm 3.$$

(ii) if $x=-\frac{72y}{23}$, $-\frac{72y^2}{23}+4y^2=115$, whence $y=\pm\frac{23}{2}$;

$$\therefore\ x=-\frac{72y}{23}=\mp 36.$$

12. $3x(x+y)=20y(x-y)$;

$\therefore\ 3x^2-17xy+20y^2=0$; *i.e.* $(3x-5y)(x-4y)=0$; $\therefore\ x=\frac{5y}{3}$, or $x=4y$.

Thus from II, (i) if $x=\frac{5y}{3}$, $y\left(\frac{2}{3}y\right)=6$, whence $y=\pm 3$; $\therefore\ x=\frac{5y}{3}=\pm 5$.

(ii) if $x=4y$, $y(3y)=6$, whence $y=\pm\sqrt{2}$;
$\therefore\ x=4y=\pm 4\sqrt{2}$.

13. $2(x^2-3xy+2y^2)=2x^2+y^2$;

$\therefore\ -6xy+3y^2=0$; *i.e.* $-3y(2x-y)=0$; $\therefore\ y=0$, or $y=2x$.

Thus from II, (i) if $y=0$, $2x^2=6$, whence $x=\pm\sqrt{3}$;

(ii) if $y=2x$, $2x^2+4x^2=6$, whence $x=\pm 1$; $\therefore\ y=2x=\pm 2$.

14. From I, $x^2-2xy=-5$; from II, $x^2-2xy+y^2=4$;

$\therefore\ 4(x^2-2xy)=-5(x^2-2xy+y^2)$;

$\therefore\ 9x^2-18xy+5y^2=0$; *i.e.* $(3x-5y)(3x-y)=0$; $\therefore\ x=\frac{5y}{3}$, or $x=\frac{y}{3}$.

Thus from I, (i) if $x=\frac{5y}{3}$, $\frac{25y^2}{9}-\frac{10y^2}{3}+5=0$, whence $y=\pm 3$;

$\therefore\ x=\frac{5y}{3}=\pm 5$.

(ii) if $x=\frac{y}{3}$, $\frac{y^2}{9}-\frac{2y^2}{3}+5=0$, whence $y=\pm 3$; $\therefore\ x=\frac{y}{3}=\pm 1$.

15. (I+II); $4x^2+17xy+4y^2=0$; *i.e.* $(4x+y)(x+4y)=0$;

$\therefore\ x=-\frac{y}{4}$, or $x=-4y$.

Thus from I, (i) if $x=-\frac{y}{4}$, $\frac{y^2}{16}+5y^2=84$, whence $y=\pm\frac{8}{9}\sqrt{21}$;

$\therefore\ x=-\frac{y}{4}=\mp\frac{2}{9}\sqrt{21}$.

(ii) if $x=-4y$, $16y^2+5y^2=84$, whence $y=\pm 2$;
$\therefore\ x=-4y=\mp 8$.

16. $5(x^2-7xy-9y^2)=9(x^2+5xy+11y^2)$;

$\therefore\ -4x^2-80xy-144y^2=0$; *i.e.* $-4(x+2y)(x+18y)=0$;

$\therefore\ x=-2y$, or $x=-18y$.

Thus from I, (i) if $x=-2y$, $4y^2+14y^2-9y^2=9$,
whence $y=\pm 1$; $\therefore\ x=-2y=\mp 2$.

(ii) if $x=-18y$, $324y^2+126y^2-9y^2=9$,
whence $y=\pm\frac{1}{7}$; $\therefore\ x=-18y=\mp\frac{18}{7}$.

17. $3(4x^2-3xy)=5(y^2-xy)$;

$\therefore\ 12x^2-4xy-5y^2=0$; *i.e.* $(6x-5y)(2x+y)=0$; $\therefore\ x=\frac{5y}{6}$, or $x=-\frac{y}{2}$.

Thus from II, (i) if $x=\frac{5y}{6}$, $y^2-\frac{5y^2}{6}=6$; whence $y=\pm 6$; $\therefore\ x=\frac{5y}{6}=\pm 5$.

(ii) if $x=-\frac{y}{2}$, $y^2+\frac{y^2}{2}=6$; whence $y=\pm 2$;

$$\therefore\ x=-\frac{y}{2}=\mp 1.$$

18. $9(x^2+3xy)=40(4y^2+xy)$;

$\therefore\ 9x^2-13xy-160y^2=0$; *i.e.* $(9x+32y)(x-5y)=0$; $\therefore\ x=-\frac{32y}{9}$, or $x=5y$.

Thus, from II, (i) if $x=-\frac{32y}{9}$, $4y^2-\frac{32y^2}{9}=9$,

whence $y=\pm\frac{9}{2}$; $\therefore\ x=-\frac{32y}{9}=\mp 16$.

(ii) if $x=5y$, $4y^2+5y^2=9$, whence $y=\pm 1$; $\therefore\ x=5y=\pm 5$.

19. $6(x^2+xy+y^2)=7(6x^2-2xy+y^2)$;

$\therefore\ 36x^2+20xy-y^2=0$; *i.e.* $-(2x-y)(18x-y)=0$; $\therefore\ y=2x$, or $y=18x$.

Thus from I, (i) if $y=2x$, $x^2+2x^2+4x^2=7$;

whence $x=\pm 1$; $\therefore\ y=2x=\pm 2$.

(ii) if $y=18x$, $x^2+18x^2+324x^2=7$;

whence $x=\pm\frac{1}{7}$; $\therefore\ y=18x=\pm\frac{18}{7}$.

20. $19(2x^2-2xy+3y^2)=18(3x^2-2y^2)$;

$\therefore\ -16x^2-38xy+93y^2=0$; *i.e.* $-(8x+31y)(2x-3y)=0$;

$$\therefore\ x=-\frac{31y}{8}, \text{ or } x=\frac{3y}{2}.$$

Thus from II, (i) if $x=-\frac{31y}{8}$, $\frac{2883y^2}{64}-2y^2=19$,

whence $y=\pm\frac{8}{\sqrt{145}}$; $\therefore\ x=-\frac{31y}{8}=\mp\frac{31}{\sqrt{145}}$.

(ii) if $x=\frac{3y}{2}$, $\frac{27y^2}{4}-2y^2=19$;

whence $y=\pm 2$; $\therefore\ x=\frac{3y}{2}=\pm 3$.

21. We have $x = -y-1$; hence $\frac{1}{y} - \frac{1}{y+1} = \frac{1}{12}$,

whence $$y^2+y-12=0; \quad \therefore\ y=3, \text{ or } -4.$$

If $$y=3,\ x=-y-1=-4;$$

and if $$y=-4,\ x=-y-1=3.$$

22. From II, $6y+6x-xy=0$; thus, since
$x=-y-3$, we obtain $6y+6(-y-3)-(-y-3)y=0$;
$\therefore\ y^2+3y-18=0$; whence $y=-6$, or 3.

If $$y=-6,\ x=-y-3=3;$$

and if $$y=3,\ x=-y-3=-6.$$

23. Multiply the first equation by 6, and the second by xy; then we have $$2x+3y=10 \text{ and } 2y+3x=0, \text{ \&c.}$$

24. From I, $$5x+2y=50,\ \therefore\ x=\frac{50-2y}{5}.$$

From II, $$12y+30x=5xy.$$

Whence $$12y+30\left(\frac{50-2y}{5}\right)=5\left(\frac{50-2y}{5}\right)y;$$

whence $2y^2-50y+300=0$; *i.e.* $2(y-10)(y-15)=0$; $\therefore\ y=10$, or 15.

If $$y=10,\ x=\frac{50-2y}{5}=6;$$

and if $$y=15,\ x=\frac{50-2y}{5}=4.$$

25. From I, $xy+x+y-9=0$. From II, $xy=3$, and $\therefore\ x=\frac{3}{y}$.

Hence $$3+\frac{3}{y}+y-9=0;$$

$\therefore\ 3y+3+y^2-9y=0$; $\therefore\ y^2-6y+3=0$, *i.e.* $(y-3)^2-6=0$; $\therefore\ y=3\pm\sqrt{6}$.

But, from I, $xy+x+y-9=0$; *i.e.* $3+x+y-9=0$;

whence $$x=6-y=6-(3\pm\sqrt{6})=3\mp\sqrt{6}.$$

26. From I, $xy+x+3y-1=0$. From II, $xy=-1$; $\therefore\ x=-\frac{1}{y}$.

Hence $-1-\frac{1}{y}+3y-1=0$; $\therefore\ -y-1+3y^2-y=0$, or $3y^2-2y-1=0$;

whence $$y=1 \text{ or } -\frac{1}{3}.$$

Then, from I, $xy+x+3y-1=0$, *i.e.* $-1+x+3y-1=0$;
$\therefore\ x=2-3y$; thus if $y=1$, $x=2-3=-1$; and if $y=-\frac{1}{3}$, $x=2+1=3$.

LI. Pages 192, 193.

1. From I, $x=3+y$; substitute in II, then $(3+y)^3-y^3=279$,
whence $9y^2+27y-252=0$; *i.e.* $9(y+7)(y-4)=0$;
$\therefore y=-7$, or 4. If $y=-7$, $x=3+y=-4$; and if $y=4$, $x=3+y=7$.

2. From I, $x=2+y$; hence $(2+y)^3-y^3=98$;
whence $6y^2+12y-90=0$; *i.e.* $6(y+5)(y-3)=0$;
$\therefore y=-5$, or 3. If $y=-5$, $x=2+y=-3$; and if $y=3$, $x=2+y=5$.

3. From I, $x=7-y$; hence $(7-y)^3+y^3=91$,
whence $21y^2-147y+252=0$; *i.e.* $21(y-3)(y-4)=0$; $\therefore y=3$, or 4.
If $y=3$, $x=7-y=4$; and if $y=4$, $x=7-y=3$.

4. From I, $x=1-y$; hence $(1-y)^3+y^3=61$,
whence $3y^2-3y-60=0$, *i.e.* $3(y-5)(y+4)=0$;
$\therefore y=5$, or -4. If $y=5$, $x=1-y=-4$; and if $y=-4$, $x=1-y=5$.

5. $\dfrac{x^4+x^2y^2+y^4}{x^2+xy+y^2}=\dfrac{91}{13}$; *i.e.* $x^2-xy+y^2=7$. (III).
(I+III); $2x^2+2y^2=20$; $\therefore x^2+y^2=10$. (IV).
(I−III); $2xy=6$. (V).
(IV+V); $(x+y)^2=16$; $\therefore x+y=\pm 4$.
(IV−V); $(x-y)^2=4$; $\therefore x-y=\pm 2$.
From $x+y=4$, $x-y=2$, we obtain $x=3$, $y=1$.
From $x+y=4$, $x-y=-2$, ... $x=1$, $y=3$.
From $x+y=-4$, $x-y=2$, ... $x=-1$, $y=-3$.
From $x+y=-4$, $x-y=-2$, ... $x=-3$, $y=-1$.

6. $\dfrac{x^4+x^2y^2+y^4}{x^2-xy+y^2}=\dfrac{243}{9}$; *i.e.* $x^2+xy+y^2=27$. (III).
(III+I); $2x^2+2y^2=36$; $\therefore x^2+y^2=18$. IV.
(III−I); $2xy=18$. V.
(IV+V); $(x+y)^2=36$; $\therefore x+y=\pm 6$. VI.
(IV−V); $(x-y)^2=0$, $\therefore x=y$, whence and from VI, $2y=\pm 6$;
$\therefore y=\pm 3$, and $\therefore x=y=\pm 3$.

7. (II+I × 3); $x^3+3x^2y+3xy^2+y^3=280+720$;
i.e $(x+y)^3=1000$; $\therefore x+y=10$, and $\therefore x=10-y$.
Substitute in I; then $10y(10-y)=240$; $\therefore -y^2+10y-24=0$;
whence $y=6$, or 4. If $y=6$, $x=10-y=4$; and if $y=4$, $x=10-y=6$.

8. (II − I × 3); $x^3 - 3x^2y + 3xy^2 - y^3 = 63 - 36$;

$$i.e.\ (x-y)^3 = 27,\ \therefore\ x - y = 3,\ \text{and}\ \therefore\ x = 3 + y.$$

Thus, from I, $3y(3+y) = 12$; $\therefore\ y^2 + 3y - 4 = 0$, whence $y = 1$, or -4.
If $y = 1$, $x = 3 + y = 4$; and if $y = -4$, $x = 3 + y = -1$.

9. We have $\left(\frac{1}{x} - \frac{1}{y}\right)^2 = 4$, *i.e.* $\frac{1}{x^2} - \frac{2}{xy} + \frac{1}{y^2} = 4$.

Hence, from II, $\frac{2}{xy} = 16$; $\therefore\ \frac{1}{x^2} + \frac{2}{xy} + \frac{1}{y^2} = 36$.

Hence $\frac{1}{x} + \frac{1}{y} = \pm 6$; which with $\frac{1}{x} - \frac{1}{y} = 2$ gives $\frac{2}{x} = 8$ or -4, and $\frac{2}{y} = 4$ or -8, &c.

10. We have $\frac{2}{x} + \frac{1}{y} = 1$ and $\left(\frac{2}{x} + \frac{1}{y}\right)\left(\frac{1}{x} + \frac{1}{y}\right) = 5$.

Hence $\frac{1}{x} + \frac{1}{y} = 5$; whence $\frac{1}{x} = -4$, and then $\frac{1}{y} = 9$.

11. $3\left(\frac{3}{x^2} - \frac{1}{y^2}\right) = \frac{5}{x^2} - \frac{1}{xy} + \frac{2}{y^2}$; $\therefore\ \frac{4}{x^2} + \frac{1}{xy} - \frac{5}{y^2} = 0$,

$$i.e.\ \left(\frac{1}{x} - \frac{1}{y}\right)\left(\frac{4}{x} + \frac{5}{y}\right) = 0.$$

If $\frac{1}{x} - \frac{1}{y} = 0$, $\frac{3}{x^2} - \frac{1}{x^2} = 1$, &c. If $\frac{4}{x} + \frac{5}{y} = 0$, $\frac{3}{x^2} - \left(\frac{4}{5x}\right)^2 = 1$, &c.

12. $3\left(\frac{1}{x^2} - \frac{1}{4y^2}\right) = \frac{1}{x^2} - \frac{1}{xy} + \frac{1}{4y^2}$; *i.e.* $\frac{2}{x^2} + \frac{1}{xy} - \frac{1}{y^2} = 0$,

$$i.e.\ \left(\frac{1}{x} + \frac{1}{y}\right)\left(\frac{2}{x} - \frac{1}{y}\right) = 0.$$

If $\frac{1}{x} + \frac{1}{y} = 0$, $\frac{1}{x^2} - \frac{1}{4x^2} = 3$, &c.

If $\frac{2}{x} - \frac{1}{y} = 0$, $\frac{1}{x^2} - \frac{1}{x^2} = 3$. [This shews that $\frac{1}{x}$ is infinite, see *Treatise on Algebra*, Arts. 122 and 123.]

13. From II, $(xy + 1)(xy + 12) = 0$; $\therefore\ xy = -1$, or $xy = -12$;

$$i.e.\ x = -\frac{1}{y},\ \text{or}\ x = -\frac{12}{y}.$$

Thus, from I, (i) if $x = -\frac{1}{y}$, $-\frac{1}{y} + y = 1$, $\therefore\ -1 + y^2 = y$;

whence $y = \frac{1}{2} \pm \frac{\sqrt{5}}{2}$; and then $x = \frac{1}{2} \mp \frac{\sqrt{5}}{2}$.

$$\text{(ii) if } x = -\frac{12}{y},\ -\frac{12}{y} + y = 1;\ \therefore\ -12 + y^2 = y;$$

whence $y = 4$, or -3. And, from I, if $y = 4$, $x = -3$; if $y = -3$, $x = 4$.

14. From II, $x^2y^2+4xy-12=0$, whence $xy=-6$, or $xy=2$; thus

$$x=-\frac{6}{y}, \text{ or } x=\frac{2}{y}.$$

(i) If $x=-\frac{6}{y}$, $-\frac{6}{y}+y=5$, $\therefore\ -6+y^2=5y$, whence $y=6$, or -1.

And, from I, if $y=6$, $x=-1$; and if $y=-1$, $x=6$.

(ii) If $x=\frac{2}{y}$, $\frac{2}{y}+y=5$, $\therefore\ 2+y^2=5y$, whence $y=\frac{5}{2}\pm\frac{\sqrt{17}}{2}$;

whence $$x=5-y=\frac{5}{2}\mp\frac{\sqrt{17}}{2}.$$

15. Add; then $x^2-2xy+y^2+y-x=6$; *i.e.* $(y-x)^2+(y-x)-6=0$; whence $(y-x)=2$ or -3; thus $y=x+2$, or $y=x-3$.

Thus, from I, (i) if $y=x+2$, $x^2-x(x+2)+x+2=-6$;

whence $$x=8;\ \therefore\ y=x+2=10.$$

And (ii), if $y=x-3$, $x^2-x(x-3)+x-3=-6$;

whence $$x=-\frac{3}{4};\ \therefore\ y=x-3=-\frac{15}{4}.$$

16. Add; then $x^2+2xy+y^2-x-y=6$; *i.e.* $(x+y)^2-(x+y)-6=0$; whence $(x+y)=3$ or $x+y=-2$; whence $y=3-x$, or $y=-2-x$.

Thus, from I, (i) if $y=3-x$, $x^2+x(3-x)-(3-x)=9$,

whence $$x=3,\ \therefore\ y=3-x=0.$$

And (ii), if $y=-2-x$, $x^2+x(-2-x)-(-2-x)=9$,

whence $$x=-7,\ \therefore\ y=-2-x=5.$$

17. Multiply by y and x respectively, and subtract; then $0=3y-\frac{4}{3}x$;

whence $$y=\frac{4x}{9}.$$

Thus, from I, $x+\frac{9}{4x}=3$; whence $4x^2-12x+9=0$, *i.e.* $(2x-3)^2=0$,

whence $$x=\frac{3}{2},\ \therefore\ y=\frac{4x}{9}=\frac{2}{3}.$$

18. $\text{I}\times y-\text{II}\times x$ gives $0=\frac{18y}{7}-\frac{7x}{4}$; whence $y=\frac{49x}{72}$.

Then, from I, $x+\frac{72}{49x}=\frac{18}{7}$; whence $49x^2-126x+72=0$;

i.e. $(7x-6)(7x-12)=0$; $\therefore\ x=\frac{6}{7}$, or $\frac{12}{7}$. Thence $y=\frac{49x}{72}=\frac{7}{12}$, or $\frac{7}{6}$.

19. $5\,(2x^2-xy+y^2)=2\,(2x^2+4xy)$; $\therefore\ 6x^2-13xy+5y^2=0$;

$$i.e.\ (2x-y)\,(3x-5y)=0;\quad \therefore\ y=2x,\ \text{or}\ \frac{3x}{5}.$$

Then, from II, (i) if $y=2x$, $2x^2+8x^2=10x$, whence $x=0$, or 1;

thence $$y=2x=0,\ \text{or}\ 2.$$

$$\text{(ii) If } y=\frac{3x}{5},\ 2x^2+\frac{12x^2}{5}=3x,\ \text{whence } x=0,\ \text{or}\ \frac{15}{22};$$

thence $$y=\frac{3x}{5}=0,\ \text{or}\ \frac{9}{22}.$$

20. $2\,(x^3+1)-3\,(x^2+x)=0$; *i.e.* $(x+1)\,\{2\,(x^2-x+1)-3x\}=0$;

$$i.e.\ (x+1)\,(2x-1)\,(x-2)=0;\quad \therefore\ x=-1,\ \frac{1}{2},\ \text{or}\ 2.$$

Then, from II, $$y=\frac{x^2+x}{6}=0,\ \frac{1}{8},\ \text{or}\ 1.$$

21. $\text{I}\times 3-\text{II}\times 2$ gives $y^3=-27$; $\therefore\ y=-3$;

whence $$2x^3-135=115;\quad \therefore\ x=5.$$

22. From II, $xy=\pm 20$. Then, from I, $x+y=\pm 9$.

Hence $$x\pm\frac{20}{x}=\pm 9;\quad \therefore\ x^2\mp 9x\pm 20=0.$$

If $x^2-9x+20=0$, $x=4$ or 5; and then $y=5$ or 4.

If $x^2+9x-20=0$, $x=-\frac{9}{2}\pm\frac{1}{2}\sqrt{161}$; and then $y=-\frac{9}{2}\mp\frac{1}{2}\sqrt{161}$.

23. $$\frac{x^4+x^2y^2+y^4}{(x^2-xy+y^2)\,(x-y)^2}=\frac{91}{28};\quad i.e.\ \frac{x^2+xy+y^2}{(x-y)^2}=\frac{13}{4}.$$

$\therefore\ 4\,(x^2+xy+y^2)=13\,(x-y)^2$; whence $-9x^2+30xy-9y^2=0$;

whence $$x=3y\ \text{or}\ 3x=y.$$

Then, from I, (i) if $x=3y$, $81y^4+9y^4+y^4=91$, $\therefore\ y^4=1$, $\therefore\ y^2=\pm 1$;

$$\therefore\ y=\pm 1\ \text{or}\ \pm\sqrt{-1};\ \text{and then}\ x=\pm 3\ \text{or}\ \pm 3\sqrt{-1}.$$

Similarly, if $y=3x$, $x=\pm 1$ or $\pm\sqrt{-1}$; and then

$$y=\pm 3\ \text{or}\ \pm 3\sqrt{-1}.$$

24. Multiply the first equation by 7 and add the second; then

$$10x^2+20xy+10y^2+30\,(x+y)=100;\quad \therefore\ x^2+2xy+y^2+3\,(x+y)=10;$$

$$\therefore\ (x+y)^2+3\,(x+y)-10=0;\ \text{whence}\ x+y=-5,\ \text{or}\ 2.$$

Now proceed as in Art. 156.

25. By subtraction, $\frac{x}{y} - \frac{y}{x} = \frac{5}{6}$;

$\therefore\ 6x^2 - 6y^2 = 5xy$, whence $(2x - 3y)(3x + 2y) = 0$, thus $x = \frac{3y}{2}$, or $x = -\frac{2y}{3}$.

Then, from I, (i) if $x = \frac{3y}{2}$, $\left(\frac{3y}{2}\right) y + \frac{3}{2} = \frac{5}{3}$,

whence $$y = \pm\frac{1}{3}, \text{ and } \therefore\ x = \frac{3y}{2} = \pm\frac{1}{2}.$$

(ii) If $x = -\frac{2y}{3}$, $\left(-\frac{2y}{3}\right) y - \frac{2}{3} = \frac{5}{3}$, whence $y^2 + \frac{7}{2} = 0$,

whence $$y = \pm\frac{\sqrt{-14}}{2};\ \therefore\ x = -\frac{2y}{3} = \mp\frac{\sqrt{-14}}{3}.$$

26. $\dfrac{xy + \dfrac{x}{y}}{\dfrac{y}{x} + \dfrac{1}{xy}} = \dfrac{\dfrac{5}{3}}{\dfrac{20}{3}}$; which reduces to $x^2 = \frac{1}{4}$; whence $x = \pm\frac{1}{2}$.

Thus, from II, (i) if $x = \frac{1}{2}$, $\frac{y}{2} + \frac{1}{2y} = \frac{5}{3}$, whence $3y^2 - 10y + 3 = 0$,

whence $$y = 3 \text{ or } \frac{1}{3}.$$

(ii) If $x = -\frac{1}{2}$, $-\frac{y}{2} - \frac{1}{2y} = \frac{5}{3}$, whence $-3y^2 - 10y - 3 = 0$,

whence $$y = -3, \text{ or } -\frac{1}{3}.$$

27. From II, $xy - ay - bx = 0$; from I, $y = \frac{ax}{b}$;

thus $$x\left(\frac{ax}{b}\right) - a\,\frac{ax}{b} - bx = 0;$$

$\therefore\ x(ax - a^2 - b^2) = 0$; $\therefore\ x = 0$, or $\frac{a^2 + b^2}{a}$. Hence $y = \frac{ax}{b} = 0$, or $\frac{a^2 + b^2}{b}$.

28. From I, $by = b^2 + a(x - a)$; and from II, $x(by) = abx + b(by)$;

whence $$x\{b^2 + a(x - a)\} = abx + b\{b^2 + a(x - a)\};$$

whence $$ax^2 + (b^2 - 2ab - a^2)x - b(b^2 - a^2) = 0;$$

$$i.e.\ \{ax - b(a - b)\}\{x - (a + b)\} = 0;$$

thus $x = \frac{b}{a}(a - b)$, or $a + b$. Substitute for y.

29. We have $x-\frac{b^2}{y}=a-b$, and $y=\frac{a^2}{x}-a+b$.

Hence $$x-\frac{b^2}{\frac{a^2}{x}-a+b}=a-b;$$

whence $$x\{a^2-(a-b)x\}-b^2x=(a-b)\{a^2-(a-b)x\};$$

$\therefore\ (a-b)x^2-2a(a-b)x+(a-b)a^2=0;\ \therefore\ x^2-2ax+a^2=0$, whence $x=a$.

Then $$y=\frac{a^2}{x}-a+b=b.$$

30. $\frac{(zx)(xy)}{yz}=\frac{9\times 16}{4}$; *i.e.* $x^2=36;\ \therefore\ x=\pm 6$.

Then, from II, $z=\frac{9}{\pm 6}=\pm\frac{3}{2}$; and from III, $y=\frac{16}{\pm 6}=\pm\frac{8}{3}$.

31. $(x^2yz)(y^2zx)(z^2xy)=6\times 12\times 18$; *i.e.* $x^4y^4z^4=6^4$;

$$\therefore\ x^2y^2z^2=\pm 6^2;\ \therefore\ xyz=\pm 6 \text{ or } \pm 6\sqrt{-1}.$$

Then we have $x=\frac{6}{xyz}=\pm 1$ or $x=\pm\frac{1}{\sqrt{-1}}=\mp\sqrt{-1}$, &c.

32. $x(x+y+z)+y(x+y+z)+z(x+y+z)=8+12+5$;

$$i.e.\ (x+y+z)^2=25;\ \therefore\ x+y+z=\pm 5;$$

$\therefore$ from I, $x=\frac{8}{x+y+z}=\frac{8}{\pm 5}=\pm\frac{8}{5}$; from II, $y=\frac{12}{\pm 5}=\pm\frac{12}{5}$;

and from III, $$z=\frac{5}{\pm 5}=\pm 1.$$

33. We have $xy+xz=6,\ yz+yx=12,\ xz+yz=10$.

Hence $$12+10-6=2yz;\ \therefore\ yz=8.$$

Then $$xz=2 \text{ and } xy=4.$$

Hence $$xz\times xy\div yz=1,\ i.e.\ x^2=1;\ \therefore\ x=\pm 1, \text{ \&c.}$$

34. By continued multiplication

$$(x+y)^2(x+z)^2(y+z)^2=2\times 3\times 6;\ \therefore\ (x+y)(x+z)(y+z)=\pm 6.$$

Hence $$y+z=\pm 3,\ x+z=\pm 2,\ x+y=\pm 1.$$

[All the upper signs or all the lower signs being taken.]

These last equations give $x=0,\ y=\pm 1,\ z=\pm 2$.

LII. Pages 197—199.

1. Let x and $3x$ be the numbers. Then $x \times 3x = 243$; $\therefore\ x^2 = 81$, &c.

2. Let x, $18 - x$ be the numbers. Then $x(18 - x) = 77$, &c.

3. Let x, $x + 20$ be the numbers. Then $x^2 + (x + 20)^2 = 650$, &c.

4. Let x and $25 - x$ be the two parts. Then $x(25 - x) = 156$, &c.

5. Let x and $80 - x$ be the two numbers. Then $x^2 + (80 - x)^2 = 3208$.

6. Let x be the number. Then $(36 - x)(30 - x) = 891$, &c.

7. Let the length be x yards and the breadth $x - 10$ yds. Then the area will be $x(x - 10)$. Hence $x(x - 10) = 1131$, whence $x = 39$ [$x = -29$ being inapplicable].

8. Let x be the number, then the reciprocal is $\frac{1}{x}$. Hence we have

$$\left(x + \frac{1}{x}\right)\left(x - \frac{1}{x}\right) = 3\tfrac{3}{4};\ \therefore\ x^2 - \frac{1}{x^2} = \frac{15}{4};$$
$$\therefore\ 4x^4 - 15x^2 - 4 = 0,\ i.e.\ (x^2 - 4)(4x^2 + 1) = 0.$$

Hence $x = \pm 2$ [the imaginary roots $\pm\frac{1}{2}\sqrt{-1}$ being inapplicable].

9. If x balls can be bought for 20*s.*, 125 balls cost $\frac{125 \times 20}{x}$ *s.* Hence $x = \frac{125 \times 20}{x}$; $\therefore\ x^2 = 2500$. Hence $x = 50$ [the negative root, -50, is inapplicable].

10. If x eggs cost 12 pence, 27 eggs will cost $\frac{12 \times 27}{x}$ pence. Hence

$$x = \frac{12 \times 27}{x};\ \therefore\ x^2 = 324.\quad \text{Hence } x = 18.$$

11. Let one cask have $2x$ gallons of water and the other x gallons of wine. Then after the transfers one cask will have $2x - 6$ gallons of water and 6 gallons of wine, and the other will have 6 gallons of water and $x - 6$ of wine. Hence $2x - 6 : 6 = 6 : x - 6$. Hence $(2x - 6)(x - 6) = 36$, &c.

12. Let x be the number of persons originally. Then each should have paid $\frac{80}{x}$ shillings, but each did pay $\frac{80}{x - 4}$. Hence $\frac{80}{x} = \frac{80}{x - 4} - 1$;

$\therefore\ 80(x - 4) = 80x - x(x - 4)$; $\therefore\ x^2 - 4x - 320 = 0$, *i.e.* $(x - 20)(x + 16) = 0$. Hence $x = 20$ [the negative root being inapplicable].

13. Let x be the number of articles; then each cost $\frac{240}{x}$ pence. Hence $\left(\frac{240}{x} + 2\right)(x - 2) = 252$; $\therefore\ 240 + 2x - \frac{480}{x} - 4 = 252$;

whence $x^2 - 8x - 240 = 0$, &c.

14. Let x be the number of shares; then each cost $\frac{1875}{x}$ pounds. Also $x-15$ were sold for £1740; hence each was sold for $\frac{1740}{x-15}$ pounds. Hence we have $\frac{1740}{x-15}=\frac{1875}{x}+4$; $\therefore$ $1740x=1875(x-15)+4x(x-15)$, whence $x=75$ [or $x=-93\frac{3}{4}$ which is inapplicable].

15. Let x be the number of minutes in which they could row the distance in still water. And let d yds. be the distance. Then their pace in still water is $\frac{d}{x}$ yds. per minute. Also, from the question it would take them $x+7$ minutes to drift down the d yds. with the stream. Thus the pace of the stream is $\frac{d}{x+7}$ yds. per minute.

$\therefore$ against the stream their pace is $\frac{d}{x}-\frac{d}{x+7}$ yds. per minute; but they row the distance against the stream in $8\frac{4}{7}$ minutes; thus their pace against the stream is $\frac{d}{8\frac{4}{7}}$ yds. per minute.

Thus we obtain the equation

$$\frac{d}{x}-\frac{d}{x+7}=\frac{d}{8\frac{4}{7}};\quad i.e.\ \frac{7d}{x(x+7)}=\frac{7d}{60};$$

whence $x(x+7)=60$; $\therefore$ $x^2+7x-60=0$; *i.e.* $(x+12)(x-5)=0$; whence, rejecting the negative root, $x=5$.

With the stream, their pace would be $\frac{d}{x}+\frac{d}{x+7}$ yds. per minute,

$$i.e.\ \frac{d}{5}+\frac{d}{12}\text{ yds. per minute, } i.e.\ \frac{17d}{60};$$

at which pace they would row d yds. in $\dfrac{d}{\frac{17d}{60}}$, *i.e.* in $\frac{60}{17}$ minutes.

16. Let speed of the current be x miles an hour. Then against the stream their pace is $8-x$ miles an hour; thus they row 8 miles in $\frac{8}{8-x}$ hrs. With the stream their pace is $8+x$ miles an hr.; $\therefore$ they row 8 miles in $\frac{8}{8+x}$ hrs. But the whole journey takes $2\frac{2}{3}$ hrs. Hence the equation

$$\frac{8}{8-x}+\frac{8}{8+x}=2\tfrac{2}{3},\ \&c.$$

17. Let the faster travel x miles an hour; then it runs 36 miles in $\frac{36}{x}$ hours. The other travels $x-15$ miles an hour; $\therefore$ it runs 36 miles in $\frac{36}{x-15}$ hours. Whence the equation $\frac{36}{x}=\frac{36}{x-15}-\frac{1}{5}$; $\therefore$ $x^2-15x-2700=0$; whence $x=60$. [The negative root is to be rejected.]

18. Suppose that he walked x miles per hour originally. Then the time actually taken, in hours, was $\frac{1}{x}+\frac{6}{x+1}$. Hence $\frac{1}{x}+\frac{6}{x+1}=\frac{7}{x}-\frac{1}{2}$; $\therefore$ $\frac{6}{x}-\frac{6}{x+1}=\frac{1}{2}$; whence $x=3$. Then the time in hours $=\frac{1}{3}+\frac{6}{4}=1\frac{5}{6}$.

19. Let A and B do the whole together in x days. Then A will do $\frac{1}{2}$ the work in $x-1$ days, and $\therefore$ A does $\frac{1}{2(x-1)}$ of the whole in a day. Similarly B does $\frac{1}{2(x+2)}$ of the whole in a day. But between them they do $\frac{1}{x}$ of the whole in a day.

Hence $\frac{1}{2(x-1)}+\frac{1}{2(x+2)}=\frac{1}{x}$; whence $x=4$.

20. Let x shillings be the original price per dozen; then $\frac{12}{x}\times 21$ could be bought for a guinea. Also $\frac{12}{x+3}\times 21$ can be bought at the new price.

Hence
$$\frac{12}{x}\times 21=\frac{12}{x+3}\times 21+7;$$
$$\therefore\ \frac{36}{x}-\frac{36}{x+3}=1;\ \therefore\ x(x+3)=108, \text{ whence } x=9.$$

21. Let the original price be x pence for 12 eggs, that is 1*d.* for $\frac{12}{x}$ eggs, or 12*d.* for $\frac{144}{x}$ eggs. At $x-1$ pence for 12 eggs, 12*d.* would buy $\frac{144}{x-1}$ eggs, and this number of eggs exceeds the former by 2.

Hence
$$\frac{144}{x-1}-\frac{144}{x}=2;\ \therefore\ 144=2x(x-1);$$
$$\therefore\ x^2-x-72=0, \textit{ i.e. } (x-9)(x+8)=0.$$
Hence
$$x=9.$$

22. Suppose she bought x dozen for 36*d.*, that is 1 dozen for $\frac{36}{x}$ *d.* If $x-1$ dozen had cost 36*d.*, 1 dozen would have cost $\frac{36}{x-1}$ *d.* Hence $\frac{36}{x-1}=\frac{36}{x}+3$, whence $x^2-x-12=0$, *i.e.* $(x-4)(x+3)=0$. Hence $x=4$.

23. Let the prices for 14 lbs. be x pence and $(x+21)$ pence. Then for 240 pence can be bought $\frac{14\times 240}{x}$ pounds and $\frac{14\times 240}{x+21}$ pounds of the two kinds respectively. Hence $\frac{14\times 240}{x}=\frac{14\times 240}{x+21}+8$,

whence $x(x+21)=21\times 420$, *i.e.* $(x-84)(x+105)=0$.
Hence $x=84$ [the negative root being inadmissible].

24. Let x apples cost 12 pence; then 1 apple costs $\frac{12}{x}$ pence, and 12 apples cost $\frac{144}{x}$ pence. Hence $\frac{144}{x}=2x+2$; $\therefore\ x^2+x-72=0$, whence $x=8$.

25. Let A's share be £x; then B's share will be £$(x-5)$. In A's share there are $20x$ shillings; thus C has $20x$ times B's share, *i.e.* £$20x(x-5)$. Whence the equation $x+(x-5)+20x(x-5)=1015$; $\therefore\ 10x^2-49x-510=0$; *i.e.* $(x-10)(10x+51)=0$; whence, $x=10$; thus A's share is £10, B's £5, and C's £1000.

26. Let x be one fraction; since their sum is $\frac{5}{6}$, the other is $\frac{5}{6}-x$; since their difference = their product, we have

$$x-\left(\frac{5}{6}-x\right)=x\left(\frac{5}{6}-x\right);$$

whence $x^2+\frac{7}{6}x-\frac{5}{6}=0$, $\therefore\ 6x^2+7x-5=0$; $\therefore\ x=\frac{1}{2}$, or $-\frac{5}{3}$.

27. The two men walk between them 25 miles in 5 hours, that is 5 miles an hour. Hence if one walk x miles per hour, the other must walk $5-x$ miles per hour; and they will walk 1 mile in $\frac{60}{x}$ and $\frac{60}{5-x}$ minutes respectively. Hence $\frac{60}{5-x}-\frac{60}{x}=18$, whence $3x^2+5x-50=0$. Hence $x=3\frac{1}{3}$ [the negative root being inapplicable], and $5-x=1\frac{2}{3}$.

28. Let the number of men in the depth of the column be x; and in its breadth $\frac{x}{2}$; thus, in the column there will be $\frac{1}{2}x^2$ men. Thus, by the question, *in the hollow square there will be* $\frac{x^2}{2}-206$ men.

By the question, the number of men in the outer side of the hollow square is x. Now if we place, in the middle of the hollow square, a solid square of men with $x-6$ in a side, we should just fill it up and thus form a solid square with x men in a side. This solid square would contain x^2 men, and the number of the men introduced to fill up the hollow square is $(x-6)^2$. Thus *the hollow square itself contains* $x^2-(x-6)^2$ *men*, *i.e.* $12x-36$.

Whence the equation $\frac{x^2}{2}-206=12x-36$;

$$\therefore\ x^2-24x-340=0;\ \textit{i.e.}\ (x-34)(x+10)=0;\ \therefore\ x=34.$$

Thus the number of men in the column is $\frac{1}{2}x^2=578$.

29. Let the lengths of its sides be x and y inches respectively. Then we have $xy=(x-6)^2$ and $xy=(x-2)(y+1)$.

The second equation reduces to $x-2y=2$; $\therefore\ y=\frac{1}{2}(x-2)$.

Hence $$x\left(\frac{x-2}{2}\right)=(x-6)^2;$$

whence $x^2-22x+72=0$; *i.e.* $(x-4)(x-18)=0$; $\therefore\ x=4$, or 18.

If $x=18$, $y=\frac{x-2}{2}=8$. [The solution $x=4$, $y=1$ must be rejected, for we could not have a square whose side was shorter than 4 inches by 6 inches.]

30. Let x, $x+35$ be the two sides; then the square on the diagonal is
$$x^2+(x+35)^2.$$

Hence $x+35+\sqrt{\{x^2+(x+35)^2\}}=5x$; $\therefore\ x^2+(x+35)^2=(4x-35)^2$; whence $x=25$, and $x+35=60$. Hence area $=60\times 25$ sq. yds.

31. Let the longer side be x yards, and the shorter y yards. Then
$$(x-3)(y-1)=\frac{1}{2}xy, \text{ and } (x+9)(y-2)=xy.$$
$$\therefore\ xy-6y-2x+6=0, \text{ and } 9y-2x=18.$$
Proceed as in Art. 156.

32. Let A, B, C, D travel at the rate of a, b, c, d miles per hour respectively; and let x be the distance from P to Q.

Then A goes 120 miles while C goes $x-120$ miles,
and A ... 140 .... D ... $x-140$ ....
Also B ... $x-126$ .... C ... 126 ......,
and B ... $\frac{x}{2}$ .... D ... $\frac{x}{2}$ ....

Hence $\frac{120}{a}=\frac{x-120}{c}$, $\frac{140}{a}=\frac{x-140}{d}$; $\therefore\ \frac{120}{140}\frac{c}{d}=\frac{x-120}{x-140}$.

Also $\frac{x-126}{b}=\frac{126}{c}$, and $b=d$; $\therefore\ \frac{d}{c}=\frac{x-126}{126}$.

Hence $$\frac{120}{140}=\frac{x-120}{x-140}\times\frac{x-126}{126},$$

whence $x^2-354x+30240=0$, *i.e.* $(x-144)(x-210)=0$.

MISCELLANEOUS EXAMPLES. IV. PAGES 199—203.

A. 1. $2x-[3x-9y-\{2x-3y-x-5y\}]=2x-[3x-9y-\{x-8y\}]$
$=2x-[3x-9y-x+8y]=2x-[2x-y]=2x-2x+y=y.$

2.
$$\begin{array}{l}
a^2+5ab-2ac+25b^2+10bc+4c^2 \\
a-5b+2c \\
\hline
a^3+5a^2b-2a^2c+25ab^2+10abc+4ac^2 \\
\quad -5a^2b-25ab^2+10abc-125b^3-50b^2c-20bc^2 \\
\quad +2a^2c+10abc-4ac^2+50b^2c+20bc^2+8c^3 \\
\hline
a^3+30abc-125b^3+8c^3
\end{array}$$

3.
$$\begin{array}{l}
x-(a-2b)\;)\; x^3+(4ab-b^2)\,x-(a-2b)\,(a^2+3b^2)\;(\;x^2+(a-2b)\,x+(a^2+3b^2) \\
\qquad x^3-(a-2b)\,x^2 \\
\hline
\qquad (a-2b)\,x^2+(4ab-b^2)\,x-(a-2b)\,(a^2+3b^2) \\
\qquad (a-2b)\,x^2-(a^2-4ab+4b^2)\,x \\
\hline
\qquad\qquad (a^2+3b^2)\,x-(a-2b)\,(a^2+3b^2) \\
\qquad\qquad (a^2+3b^2)\,x-(a-2b)\,(a^2+3b^2)
\end{array}$$

4. (i) $\{(2x+y-z)-(x+2y+4z)\}\{(2x+y-z)+(x+2y+4z)\}$

$=\{x-y-5z\}\{3x+3y+3z\}=3\{x-y-5z\}(x+y+z)$.

(ii) $x^2(y^2-1)-(y^2-1)=(x^2-1)(y^2-1)=(x-1)(x+1)(y-1)(y+1)$.

(iii) $x^2z(y^2z-1)-(y^2z-1)=(x^2z-1)(y^2z-1)$.

5. (i) Find the H.C.F.

$$\begin{array}{l}
x^3+4x^2-8x+24\;)\; x^4-x^3+8x-8\;(\;x-5 \\
\qquad x^4+4x^3-8x^2+24x \\
\hline
\qquad -5x^3+8x^2-16x-8 \\
\qquad -5x^3-20x^2+40x-120 \\
\hline
\qquad 28\;)\;28x^2-56x+112 \\
\hline
\qquad\qquad x^2-2x+4\;)\; x^3+4x^2-8x+24\;(\;x+6 \\
\qquad\qquad\qquad x^3-2x^2+4x \\
\hline
\qquad\qquad\qquad 6x^2-12x+24 \\
\qquad\qquad\qquad 6x^2-12x+24
\end{array}$$

Whence, by division, we find

$$\frac{x^3+4x^2-8x+24}{8-8x+x^3-x^4}=\frac{(x+6)(x^2-2x+4)}{(2-x-x^2)(x^2-2x+4)}=\&c.$$

(ii)
$$\frac{(x-1)^2-2(x-2)^2+(x-3)^2}{(x-1)(x-2)(x-3)}=\frac{x^2-2x+1-2x^2+8x-8+x^2-6x+9}{(x-1)(x-2)(x-3)}$$
$$=\frac{2}{(x-1)(x-2)(x-3)}.$$

6. (i) Multiply by $(x-a)(x-b)(x-c)$; then

$$(x-b)(x-c)+2(x-a)(x-c)=3(x-a)(x-b);$$

i.e. $x^2-bx-cx+bc+2x^2-2ax-2cx+2ac=3x^2-3ax-3bx+3ab$;

whence $ax+2bx-3cx=3ab-2ac-bc$; whence $x=\dfrac{3ab-2ac-bc}{a+2b-3c}$.

(ii) By inspection, $(4x+3)(x-7)=0$, &c.

(iii) $x+y=\frac{1}{x}+\frac{1}{y}=\frac{x+y}{xy}$; $\therefore xy=1$. Also $x+y=\frac{5}{2}$.

Hence $x+\frac{1}{x}=\frac{5}{2}$. Whence $x=2$ or $\frac{1}{2}$, and then $y=\frac{1}{2}$ or 2.

7. $x^2-5x+7=\left(x-\frac{5}{2}\right)^2+7-\left(\frac{5}{2}\right)^2=\left(x-\frac{5}{2}\right)^2+\frac{3}{4}$, which can never be less than $\frac{3}{4}$, since $\left(x-\frac{5}{2}\right)^2$ is either zero or a *positive* quantity.

8. Let x, and $x+1$ be the numbers. Then $(x+1)^3-x^3=919$;

whence $3x^2+3x-918=0$, *i.e.* $3(x+18)(x-17)=0$.

Hence, rejecting the negative root, $x=17$.

B. 1. $x^3+9x^2+27x+27-3(x^3+6x^2+12x+8)+3(x^3+3x^2+3x+1)-x^3=6$.

2. $(x-a)^2+(y-b)^2+(a^2+b^2-1)(x^2+y^2-1)$

$=x^2-2ax+a^2+y^2-2by+b^2+(a^2+b^2-1)x^2+(a^2+b^2-1)y^2-(a^2+b^2-1)$

$=x^2(a^2+b^2)+y^2(a^2+b^2)-2ax-2by+1.$

Again, $(ax+by-1)^2+(bx-ay)^2$

$=a^2x^2+b^2y^2+1+2abxy-2ax-2by+b^2x^2-2abxy+a^2y^2$

$=x^2(a^2+b^2)+y^2(a^2+b^2)-2ax-2by+1.$

3.

$$\begin{array}{r|rrrrrrr|l}
x^2-2ax+a^2\;) & x^6 & & & -2a^3x^3 & & & +a^6 & (\;x^4+2ax^3+3a^2x^2+2a^3x+a^4 \\
 & x^6 & -2ax^5 & +a^2x^4 & & & & & \\
\cline{2-8}
 & & 2ax^5 & -a^2x^4 & -2a^3x^3 & & & +a^6 & \\
 & & 2ax^5 & -4a^2x^4 & +2a^3x^3 & & & & \\
\cline{3-8}
 & & & 3a^2x^4 & -4a^3x^3 & & & +a^6 & \\
 & & & 3a^2x^4 & -6a^3x^3 & +3a^4x^2 & & & \\
\cline{4-8}
 & & & & 2a^3x^3 & -3a^4x^2 & & +a^6 & \\
 & & & & 2a^3x^3 & -4a^4x^2 & +2a^5x & & \\
\cline{5-8}
 & & & & & a^4x^2 & -2a^5x & +a^6 & \\
 & & & & & a^4x^2 & -2a^5x & +a^6 &
\end{array}$$

Or thus: $(x^3-a^3)^2\div(x-a)^2=\{(x^3-a^3)\div(x-a)\}^2=(x^2+ax+a^2)^2=$&c.

4. $8x^3+27=(2x)^3+3^3=(2x+3)\{(2x)^2-3(2x)+3^2\}=(2x+3)(4x^2-6x+9)$;

$16x^4+36x^2+81=(4x^2+9)^2-36x^2=\{(4x^2+9)-6x\}\{(4x^2+9)+6x\}$

$=(4x^2-6x+9)(4x^2+6x+9)$;

$6x^2-5x-6=(2x-3)(3x+2)$.

Thus L.C.M. $=(2x+3)(4x^2-6x+9)(4x^2+6x+9)(2x-3)(3x+2)$

$=(8x^3+27)(8x^3-27)(3x+2)=(64x^6-729)(3x+2)$.

5. (i) $\dfrac{2}{x-1}-\dfrac{1}{x+1}=\dfrac{2(x+1)-(x-1)}{(x-1)(x+1)}=\dfrac{x+3}{x^2-1}$.

Thus $\dfrac{2}{x-1}-\dfrac{1}{x+1}-\dfrac{x+3}{x^2+1}=\dfrac{x+3}{x^2-1}-\dfrac{x+3}{x^2+1}$

$$=\frac{(x+3)(x^2+1)-(x+3)(x^2-1)}{(x^2-1)(x^2+1)}=\frac{2(x+3)}{x^4-1}.$$

(ii) $\dfrac{\{(a+b)^2+(a+b)c+c^2\}\{(a+b)-c\}\{(a+b)+c\}}{\{(a+b)-c\}\{(a+b)^2+(a+b)c+c^2\}\{(a+b)+c\}}=1.$

6. (i) $6+11x-2x^2+3x^2+15x=x^2+2x+1-x$; $\therefore\ 25x=-5$;

$$\therefore\ x=-\frac{1}{5}.$$

(ii) $5x^2+7x-160=0$; *i.e.* $(5x+32)(x-5)=0$, &c.

(iii) $3(x^2+xy)=10(y^2-xy)$; whence $3x^2+13xy-10y^2=0$;

i.e. $(3x-2y)(x+5y)=0$; $\therefore\ x=\dfrac{2y}{3}$, or $x=-5y$.

From II, if $x=\dfrac{2y}{3}$, $y^2-\dfrac{2y^2}{3}=3$; $\therefore\ 3y^2-2y^2=9$; whence $y=\pm 3$;

and $$\therefore\ x=\frac{2y}{3}=\pm 2.$$

If $x=-5y$, $y^2+5y^2=3$; whence $y=\pm\dfrac{1}{\sqrt{2}}$; and $\therefore\ x=-5y=\mp\dfrac{5}{\sqrt{2}}$.

7. If α and β are the roots, $\alpha+\beta=5$, $\alpha\beta=2$; (§ 150)

thus $$\alpha^2+\beta^2=(\alpha+\beta)^2-2\alpha\beta=25-4=21.$$

8. Let x and $x+360$ be the number of tickets for reserved and unreserved seats respectively. Then the respective costs of a ticket are $\dfrac{1200}{x}$ and $\dfrac{1200}{x+360}$. Hence $\dfrac{1200}{x}=\dfrac{1200}{x+360}+3$;

$$\therefore\ x^2+360x-144000=0,\ \textit{i.e.}\ (x-240)(x+600)=0.$$

Hence $x=240$ [the negative root being inadmissible]; and the total number of tickets sold $=240+600$.

C. 1. $12a-3\{b-2a+6b-2a\}=12a-3\{7b-4a\}$

$$=12a-21b+12a=24a-21b.$$

2. $\{(a^3-ab^2)+(2a^2b+2b^3)\}\{(a^3-ab^2)-(2a^2b+2b^3)\}$

$$=(a^3-ab^2)^2-(2a^2b+2b^3)^2=(a^6-2a^4b^2+a^2b^4)-(4a^4b^2+8a^2b^4+4b^6)$$

$$=a^6-6a^4b^2-7a^2b^4-4b^6.$$

3.
$$\begin{array}{rrrrrrl}
-4x^2-xy+2y^2\,)\!\! & 24x^4 & -10x^3y & -8x^2y^2 & +10xy^3 & -4y^4 & (\,-6x^2+4xy-2y^2 \\
 & 24x^4 & +6x^3y & -12x^2y^2 & & & \\ \cline{2-6}
 & & -16x^3y & +4x^2y^2 & +10xy^3 & -4y^4 & \\
 & & -16x^3y & -4x^2y^2 & +8xy^3 & & \\ \cline{3-6}
 & & & 8x^2y^2 & +2xy^3 & -4y^4 & \\
 & & & 8x^2y^2 & +2xy^3 & -4y^4 &
\end{array}$$

4. $4(ab-cd)^2-(a^2+b^2-c^2-d^2)^2$
$=\{2(ab-cd)-(a^2+b^2-c^2-d^2)\}\{2(ab-cd)+(a^2+b^2-c^2-d^2)\}$
$=\{(c^2-2cd+d^2)-(a^2-2ab+b^2)\}\{(a^2+2ab+b^2)-(c^2+2cd+d^2)\}$
$=\{(c-d)^2-(a-b)^2\}\{(a+b)^2-(c+d)^2\}$
$=\{(c-d)-(a-b)\}\{(c-d)+(a-b)\}\{(a+b)-(c+d)\}\{(a+b)+(c+d)\}$
$=$&c.

5. (i)
$$\frac{x+\dfrac{y-x}{1+xy}}{1-\dfrac{x(y-x)}{1+xy}}=\frac{\left\{x+\dfrac{y-x}{1+xy}\right\}(1+xy)}{\left\{1-\dfrac{x(y-x)}{1+xy}\right\}(1+xy)}=\frac{x(1+xy)+(y-x)}{(1+xy)-x(y-x)}$$
$$=\frac{x^2y+y}{1+x^2}=\frac{(x^2+1)y}{x^2+1}=y.$$

(ii) Expression on the left
$$=\frac{1}{abc}[bc(b-c)+ca(c-a)+ab(a-b)+(b-c)(c-a)(a-b)]$$
$$=\frac{1}{abc}[b^2c-bc^2+c^2a-ca^2+a^2b-ab^2+bc^2-b^2c+ca^2-c^2a+ab^2-a^2b]=0.$$

6. (i) Multiply by 30; then $10(x-1)-6(x-3)=210+5(x+2)$;
whence $10x-6x-5x=210+10+10-18$; $\therefore\ x=-212$.

(ii) $x^2+(2a-x)^2=2a^2$; $\therefore\ x^2-2ax+a^2=0$; $\therefore\ x=a$. Then
$y=2a-x=a$.

7. $x^2+6x+12=(x+3)^2+3$; least value of $(x+3)^2$ is 0; thus least value of $x^2+6x+12$ is 3.

Again $6x-x^2-4=5-(x-3)^2$; least value of $(x-3)^2$ is 0; thus the greatest value of $5-(x-3)^2$ is 5.

8. Let A have $3x$ shillings and x pennies; and let B have y shillings and y pennies.

Then $3x+x+y+y=18$, or $2x+y=9$.
Also $3x\times 12+x=12y+y+9$, or $37x-13y=9$.
Whence $x=2$, $y=5$.

D. 1. $\{(b+c)^2-a^2\}+\{(c+a)^2-b^2\}+\{(a+b)^2-c^2\}$
$=(b+c-a)(b+c+a)+(c+a-b)(c+a+b)+(a+b-c)(a+b+c)$
$=(a+b+c)\{(b+c-a)+(c+a-b)+(a+b-c)\}=(a+b+c)(a+b+c)$.

2. $1+4x+6x^2+4x^3+x^4+2-2x+2x^2=$ &c.

3. Let the numbers be x and $x+1$. Then we have to prove that

$$(x+1)^2-x^2=2x+1.$$

4. If $\frac{1}{a}+\frac{1}{b}+\frac{1}{c}=0$; $\therefore\ bc+ca+ab=0$.

Now $(a+b+c)^2=a^2+b^2+c^2+2(bc+ca+ab)=a^2+b^2+c^2$,
since $bc+ca+ab=0$.

5. $x^2-5x-14=(x-7)(x+2)$, $(x^2-4x-21)=(x-7)(x+3)$.

The factors of the third expression are not obvious; we therefore proceed to find the H. C. F. of it and the second expression. We shall thus be led to find that $x^3-3x^2-25x-21=(x^2-4x-21)(x+1)=(x-7)(x+3)(x+1)$.

The L. C. M. is therefore $(x-7)(x+3)(x+2)(x+1)$.

All three expressions vanish if $x-7=0$.

6. (i) $\frac{4}{x}-\frac{3}{y}+5=10$, and $\frac{6}{x}+\frac{3}{y}=10$.

Hence, by addition $\frac{10}{x}=15$, $\therefore\ x=\frac{2}{3}$, &c.

(ii) $\frac{1}{x}-\frac{1}{a}=\frac{1}{x-a-b}+\frac{1}{b}$; *i.e.* $\frac{a-x}{ax}=\frac{x-a}{b(x-a-b)}$;

$\therefore\ b(x-a-b)(a-x)+ax(a-x)=0$; *i.e.* $(a-x)\{b(x-a-b)+ax\}=0$;
or $(a-x)\{x(a+b)-b(a+b)\}=0$; or $(a+b)(a-x)(x-b)=0$; thus $x=a$, or b.

(iii) $3x^2-11(3-x)^2=1$; whence $-8x^2+66x-100=0$;

$\therefore\ x=2$, or $\frac{25}{4}$. Then if $x=2$, $y=3-x=1$; if $x=\frac{25}{4}$, $y=3-x=-\frac{13}{4}$.

7. $$\frac{x_1}{x_2}+\frac{x_2}{x_1}=\frac{x_1^2+x_2^2}{x_1x_2}=\frac{(x_1+x_2)^2-2x_1x_2}{x_1x_2}=\frac{\left(-\frac{b}{a}\right)^2-2\frac{c}{a}}{\frac{c}{a}}=\frac{b^2-2ac}{ac}.$$

8. Let x gallons be drawn off, then after filling up there will be $60-x$ gallons of alcohol and x of water. We now draw off $x+14$ gallons, of which $(x+14)\times\frac{60-x}{60}$ are alcohol. There are 30 gallons of alcohol drawn off in the two drawings.

Hence $x+(x+14)\times\frac{60-x}{60}=30$; whence $x^2-106x+960=0$,

$$i.e.\ (x-10)(x-96)=0.$$

Hence $x=10$ [the value $x=96$ being clearly inapplicable].

E. 1. $$\frac{13}{6}-\left\{\frac{\frac{17}{3}}{3}-\frac{12}{4}\right\}\div\frac{10}{6}=\frac{13}{6}-\left(-\frac{10}{9}\right)\times\frac{3}{5}=\frac{13}{6}+\frac{2}{3}.$$

2. (i) $(a+b)^4-(a^2-b^2)^2=(a+b)^4-\{(a-b)(a+b)\}^2$
$=(a+b)^4-(a-b)^2(a+b)^2=(a+b)^2\{(a+b)^2-(a-b)^2\}=(a+b)^2\{4ab\}$.

(ii) Obviously $(b-c)+(c-a)+(a-b)=0$; thus by squaring,
$(b-c)^2+(c-a)^2+(a-b)^2+2(b-c)(c-a)+2(b-c)(a-b)+2(c-a)(a-b)=0$;
$\therefore (b-c)^2+(c-a)^2+(a-b)^2-2(c-b)(c-a)-2(b-c)(b-a)-2(a-c)(a-b)=0$;
$\therefore (b-c)^2+(c-a)^2+(a-b)^2=2(c-b)(c-a)+2(b-c)(b-a)+2(a-c)(a-b)$.

3. $\{(a+2b-3c+d)^2-(2a+b+3c-d)^2\}\div(a+b)$
$=\{(a+2b-3c+d)-(2a+b+3c-d)\}\{(a+2b-3c+d)+(2a+b+3c-d)\}$
$\div(a+b)$
$=\{-a+b-6c+2d\}\{3a+3b\}\div(a+b)=3\{-a+b-6c+2d\}$.

4. First find H. C. F.

$$\begin{array}{l}
6x^2-5ax-6a^2\ )\ 4x^3-\ 2ax^2-\ 9a^3\\
\hphantom{6x^2-5ax-6a^2\ )\ 4x^3-\ 2ax^2-\ }3\\
\hline
12x^3-\ 6ax^2-27a^3\ (\ 2x\\
12x^3-10ax^2-12a^2x\\
\hline
4ax^2+12a^2x-27a^3\\
\hphantom{4ax^2+12a^2x-2}3\\
\hline
12ax^2+36a^2x-81a^3\ (\ 2a\\
12ax^2-10a^2x-12a^3\\
\hline
23a^2\ )\ 46a^2x-69a^3\\
\hline
2x-\ 3a\ )\ 6x^2-5ax-6a^2\ (\ 3x+2a\\
6x^2-9ax\\
\hline
4ax-6a^2\\
4ax-6a^2
\end{array}$$

Thus (§ 121, (ii)), L. C. M.
$=(4x^3-2ax^2-9a^3)(6x^2-5ax-6a^2)\div(2x-3a)$
$=(4x^3-2ax^2-9a^3)(3x+2a)=$&c.

5. $$\frac{y^3+x^3}{xy}\times\frac{1}{y^2-x^2}+\frac{x}{xy-y^2}-\frac{y}{x^2+xy}=\frac{-(y^3+x^3)+x^2(x+y)-y^2(x-y)}{xy(x-y)(x+y)}$$
$$=\frac{x^2y-xy^2}{xy(x-y)(x+y)}=\frac{xy(x-y)}{xy(x-y)(x+y)}=\frac{1}{x+y}.$$

6. (i) $(x^2-a^2)+(x^2-b^2)=2(x-b)(x-a)$;
whence $2bx+2ax=a^2+b^2+2ab$; *i.e.* $2x(a+b)=(a+b)^2$;
whence $x=\frac{1}{2}(a+b)$.

(ii) $x^2-3x+2+x^2-5x+6+x^2-4x+3=11$;
$\therefore 3x^2-12x=0$; $\therefore 3x(x-4)=0$, &c.

(iii) Multiply by y, x respectively; then
$$xy+1=\frac{21}{10}y \text{ and } xy+1=\frac{7}{3}x.$$
Hence $$\frac{21}{10}y=\frac{7}{3}x;\ \therefore x=\frac{9}{10}y.$$

Hence $\frac{9}{10}y^2+1=\frac{21}{10}y$; whence $y=\frac{2}{3}$ or $\frac{5}{3}$, &c.

7. If $\frac{x}{a}+\frac{a}{x}=\frac{c}{a}+\frac{a}{c}$; $\frac{x}{a}-\frac{c}{a}+\frac{a}{x}-\frac{a}{c}=0$; $\therefore\ \frac{x-c}{a}-\frac{a}{cx}(x-c)=0$,

$$i.e.\ (x-c)\left(\frac{1}{a}-\frac{a}{cx}\right)=0. \text{ Hence } x=c \text{ or } x=\frac{a^2}{c}.$$

8. If he rode x miles an hour, the time actually taken was $\frac{180}{x}$ hours. The time at $x-3$ miles an hour would have been $\frac{180}{x-3}$. Hence

$$\frac{180}{x}=\frac{180}{x-3}-3, \text{ whence } x^2-3x-180=0.$$

Hence $x=15$ [the negative value of x being inapplicable].

F. 1.

$$\begin{array}{rrrr}
3a^3 & -4a^2b & +2ab^2 & \\
 & 3a^2b & -4ab^2 & +2b^3\\
2a^3 & & +3ab^2 & -4b^3\\
-4a^3 & +2a^2b & & +3b^3\\
\hline
a^3 & +a^2b & +ab^2 & +b^3
\end{array}$$

2. Both expressions are equal to

$$(b^2+c^2)\,x^2+(a^2+c^2)\,y^2+(a^2+b^2)\,z^2-2abxy-2acxz-2bcyz.$$

3.

$$\begin{array}{r r l}
x^2+3x+2\;) & -x^4+x^3+11x^2+11x+2 & (\;-x^2+4x+1\\
 & \underline{-x^4-3x^3-2x^2\qquad\qquad} & \\
 & 4x^3+13x^2+11x+2 & \\
 & \underline{4x^3+12x^2+8x\quad\;} & \\
 & x^2+3x+2 & \\
 & x^2+3x+2 &
\end{array}$$

4.

$$\begin{array}{r l}
x^3-x^2-2x+2\;) & x^4-3x^3+2x^2+x-1\;(\;x-2\\
 & \underline{x^4-x^3-2x^2+2x}\\
 & -2x^3+4x^2-x-1\\
 & \underline{-2x^3+2x^2+4x-4}\\
 & \qquad 2x^2-5x+3\;)\;x^3-x^2-2x+2\\
 & \qquad\qquad\qquad\qquad\qquad\qquad 2\\
 & \qquad\qquad\qquad\quad \overline{2x^3-2x^2-4x+4}\;(\;x\\
 & \qquad\qquad\qquad\quad \underline{2x^3-5x^2+3x\qquad}\\
 & \qquad\qquad\qquad\qquad\quad 3x^2-7x+4\\
 & \qquad\qquad\qquad\qquad\qquad\qquad\quad 2\\
 & \qquad\qquad\qquad\qquad\quad \overline{6x^2-14x+8}\;(\;3\\
 & \qquad\qquad\qquad\qquad\quad \underline{6x^2-15x+9}\\
 & \qquad\qquad\qquad\qquad\qquad\quad x-1\;)\;2x^2-5x+3\;(\;2x-3\\
 & \qquad\qquad\qquad\qquad\qquad\qquad\qquad\;\; \underline{2x^2-2x\qquad}\\
 & \qquad\qquad\qquad\qquad\qquad\qquad\qquad\quad -3x+3\\
 & \qquad\qquad\qquad\qquad\qquad\qquad\qquad\quad -3x+3
\end{array}$$

$\therefore$ H. C. F. $=x-1$.

Since $(x-1)$ is a factor of both, they will both vanish if $x-1=0$, *i.e.* if $x=1$.

5. (i) $$\frac{2}{x(x-2)} + \frac{1}{(x-2)(x-3)} - \frac{3}{x(x-3)} = \frac{2(x-3)+x-3(x-2)}{x(x-2)(x-3)}$$
$$= \frac{2x+x-3x-6+6}{x(x-2)(x+3)} = 0.$$

(ii) $$\frac{1}{1-\dfrac{1-x}{\left(1-\dfrac{1}{1-x}\right)(1-x)}} = \frac{1}{1-\dfrac{1-x}{-x}} = \frac{1}{1+\dfrac{1-x}{x}} = x.$$

6. (i) Multiply by 10; then $2(x+3)-(6-x)=10x-7$; whence $x=1$.

(ii) From the first equation

$(x+y)(a-b)+(x-y)(a+b)=2(a+b)(a-b)$; $\therefore\ ax-by=a^2-b^2$.
Also $ax+by=a^2+b^2$. Whence $x=a$, $y=b$.

(iii) We have $1+\dfrac{2}{x-1}+1+\dfrac{4}{x-2}=2\left\{1+\dfrac{6}{x-3}\right\}$;

$$\therefore\ \frac{1}{x-1}+\frac{2}{x-2}=\frac{6}{x-3};$$

$$\therefore\ (x-2)(x-3)+2(x-1)(x-3)=6(x-1)(x-2),$$

whence $$3x^2-5x=0;\ \therefore\ x=0 \text{ or } x=\frac{5}{3}.$$

7. Let α and β be the roots; then $\alpha+\beta=7$, $\alpha\beta=9$;

$$\therefore\ (\alpha-\beta)^2=(\alpha+\beta)^2-4\alpha\beta=49-36=13;\ \therefore\ \alpha-\beta=\sqrt{13};$$
$$\therefore\ \alpha^2-\beta^2=(\alpha+\beta)(\alpha-\beta)=7\sqrt{13}.$$

8. Let x^2 be the number; then its square root is x; $\therefore\ x^2-x=156$;
whence $$x=13, \text{ or } -12;\ \therefore\ x^2=169 \text{ or } 144.$$

N.B. 144 cannot exceed its square root by 156, if the *arithmetical* square root is meant.

MISCELLANEOUS EQUATIONS. Pages 203—209.

1. $x^2-5x+4+x^2-8x+15=2x^2-12x+18$; whence $x=1$.

2. Multiply by 12; then $6(x-2)-4(x-4)+3(x-5)=0$;
$$\therefore\ 6x-12-4x+16+3x-15=0;\ \therefore\ 5x=11, \text{ \&c.}$$

3. $$\frac{(x-5)-(x-3)}{(x-3)(x-5)}=\frac{(x-4)-(x-2)}{(x-2)(x-4)};\ \therefore\ \frac{-2}{(x-3)(x-5)}=\frac{-2}{(x-2)(x-4)};$$
$$\therefore\ (x-3)(x-5)=(x-2)(x-4);\ \therefore\ x^2-8x+15=x^2-6x+8;$$
whence $$-2x=-7.$$

4. By addition, $32x+32y=160$; $\therefore x+y=5$.
By subtraction, $2x-2y=2$; $\therefore x-y=1$.
Hence $2x=6$; $\therefore x=3$, and $2y=4$; $\therefore y=2$.

5. Let the first road be x miles, then the second is $x+3$.

x miles at 8 miles an hour takes him $\frac{x}{8}$ hours.

Thus, from the question, in returning he takes $7\frac{1}{2}$ minutes $\left(i.e.\ \frac{1}{8}\text{hour}\right)$ longer than this; *i.e.* he returns in $\frac{x}{8}+\frac{1}{8}$ hours.

But $x+3$ miles at 9 miles an hour, takes him $\frac{x+3}{9}$ hours; thus he returns in $\frac{x+3}{9}$ hours.

Whence, $\frac{x}{8}+\frac{1}{8}=\frac{x+3}{9}$; $\therefore 9x+9=8x+24$; whence $x=15$.

6. Multiply by 18. $9(x-2)-2(x-5)+15(x-1)=0$;
$\therefore 9x-18-2x+10+15x-15=0$; whence $22x=23$, &c.

7. $3x^2-6x-9+3x^2+8x-91=6x^2-21x+15$;
whence $23x=115$; $\therefore x=5$.

8. Multiply by $(x+2)(x+3)(x+4)$;

$$(2x+9)(x+3)(x+4)+(3x-2)(x+2)(x+4)=(5x+14)(x+2)(x+3);$$

i.e. $2x^3+23x^2+87x+108+3x^3+16x^2+12x-16=5x^3+39x^2+100x+84$;

$$\therefore 2x^3+3x^3-5x^3+23x^2+16x^2-39x^2+87x+12x-100x=84-108+16;$$

$$\therefore x=8.$$

Or thus: we have $2+\frac{5}{x+2}+3-\frac{11}{x+3}=5-\frac{6}{x+4}$;

$$\therefore \frac{5}{x+2}-\frac{5}{x+3}=\frac{6}{x+3}-\frac{6}{x+4};\quad \therefore \frac{5}{(x+2)(x+3)}=\frac{6}{(x+3)(x+4)};$$

$$\therefore 5(x+4)=6(x+2),\ \text{whence } x=8.$$

9. Clear of fractions; then $8x+7y=4592$, and $7x+8y=4648$.
By addition, $15x+15y=9240$; $\therefore x+y=616$.
By subtraction, $x-y=-56$.
Hence $2x=560$, $2y=672$.

10. Let the original cost per pound of tea and sugar be x pence and y pence respectively. Then, after the rise, the prices would be

$$\frac{120}{100}x \text{ and } \frac{125}{100}y.$$

Hence $x+5y=45$, and $\frac{120}{100}x+\frac{125}{100}\times 5y=55$, or $24x+125y=1100$.

Multiply the first equation by 25 and subtract the second; then $x=25$, &c.

11. Multiply by $9 \times 11 \times 13$; then

$$143(6x+3)+117(7x+6)+99(9x+2)=1287(2x+1);$$

whence $$858x+819x+891x-2574x=1287-429-702-198;$$

$$i.e.\ -6x=-42;\ \therefore\ x=7.$$

12. Multiply by ab; $bx+ax=a^2-b^2$; $\therefore\ x=\dfrac{a^2-b^2}{b+a}=a-b.$

13. $$\frac{x+3}{x+1}-\frac{x+4}{x+2}=\frac{x-5}{x-3}-\frac{x-6}{x-4};$$

$$\therefore\ \frac{(x+3)(x+2)-(x+4)(x+1)}{(x+1)(x+2)}=\frac{(x-5)(x-4)-(x-6)(x-3)}{(x-3)(x-4)};$$

$$i.e.\ \frac{2}{(x+1)(x+2)}=\frac{2}{(x-3)(x-4)};\ \therefore\ (x+1)(x+2)=(x-3)(x-4);$$

$$i.e.\ x^2+3x+2=x^2-7x+12;\ \text{whence } x=1.$$

14. Multiply the first equation by $a+b$, and subtract the second; then $2ay=2a$; $\therefore\ y=1$; whence $x=1$.

15. Suppose he had x eggs in May; then the daily average in May was $\dfrac{x}{31}$.

He had 100 less in April, *i.e.* $x-100$; thus his daily average in April was $\dfrac{x-100}{30}$.

Hence $\dfrac{x}{31}=\dfrac{x-100}{30}+3$; $\therefore\ 30x=31(x-100)+2790$; whence $x=310$.

16. $3x-3[4x-4x+10]=9-2[3x-5x+25]$;
$\therefore\ 3x-30=9-2[-2x+25]$; $\therefore\ 3x-30=9+4x-50$; whence $x=11$.

17. Multiply the first equation by 2, and add to the second; then

$$\frac{14}{x}=7;\ \text{whence } x=2.\quad \text{Then } \frac{12}{y}=3-\frac{4}{x}=3-2;\ \therefore\ y=12.$$

18. $(5x+3)(3x+5)=0$, &c.

19. $(x+2)(x-5)-(x-3)(x-4)=(x-4)(x-5)$;

$$i.e.\ x^2-3x-10-x^2+7x-12=x^2-9x+20;$$

$$\therefore\ x^2-13x+42=0;\ i.e.\ (x-6)(x-7)=0, \text{ \&c.}$$

20. Let the age of the youngest son be x years; then that of the oldest is $x+5$; and $\therefore$ in 7 years will be $x+12$. Thus, by the question, the father's age in 7 years will be $2(x+12)$.

But the father's age is three times that of his younger son, *i.e.* $3x$; and hence in 7 years his age will be $3x+7$.

Thus $$2(x+12)=3x+7;\ \text{whence } x=17, \text{ \&c.}$$

21. Multiply by 6; $3(x+1)(x+3)=2(x+5)(x+2)+(x-1)(x-4)$;
$\therefore 3x^2+12x+9=2x^2+14x+20+x^2-5x+4$; whence $3x=15$, &c.

22. $(b+c)x-ab-ac-(c+a)x+bc+ab=(a+b)x-ac-bc$;
$\therefore (b+c)x-(c+a)x-(a+b)x=-ac-bc+ab+ac-bc-ab$;
$\therefore -2ax=-2bc$; $\therefore x=bc/a$.

23. Subtract; then $10x-10y=20$; $\therefore x-y=2$.
Multiply last equation by 19, and subtract the first; then
$2x=34$; $\therefore x=17$. Hence $y=x-2=15$.

24. $(2x-3)(2x+1)+(3x+8)(x-3)=5(x-3)(2x+1)$;
i.e. $4x^2-4x-3+3x^2-x-24=10x^2-25x-15$;
whence $3x^2-20x+12=0$; *i.e.* $(3x-2)(x-6)=0$, &c.

25. Let x lbs. be the weight allowed free. Thus for the two passengers $2x$ lbs. are allowed free; and they have to pay 40*d.* for the remainder (viz. $360-2x$ lbs.); thus they pay at the rate of $\frac{40}{360-2x}$ pence per lb.

If one passenger take the luggage, x lbs. would be allowed free; and he would have to pay 65*d.* for the remainder (viz. $360-x$ lbs.). Thus he pays at the rate of $\frac{65}{360-x}$ pence per lb.

Whence $$\frac{40}{360-2x}=\frac{65}{360-x};$$
$\therefore 40(360-x)=65(360-2x)$; whence $90x=9000$; $\therefore x=100$.

26. Multiply by 120; then
$24(3x-1)+20(5x+1)=15(9x+1)-40(1-x)$;
$\therefore 72x-24+100x+20=135x+15-40+40x$; whence $x=7$.

27. $x+y=6x-3y-6$; whence $-5x+4y=-6$.
$2y-x=2y-2x+10$; whence $x=10$.
Then $4y=5x-6=44$.

28. By addition, $2ax+2ay=2a^2$; $\therefore x+y=a$.
By subtraction $2bx-2by=-2b^2$, $\therefore x-y=-b$.
Hence $2x=a-b$, and $2y=a+b$.

29. $3(8-x)+8(3-x)=3(3-x)(8-x)$;
i.e. $24-3x+24-8x=72-33x+3x^2$;
whence $3x^2-22x+24=0$; $\therefore (3x-4)(x-6)=0$, &c.

30. Let x be the father's present age in years, then x is also the sum of the ages of his five children. Five years ago the father's age was $x-5$, and the sum of the ages of his children was $x-25$. Hence $x-5=2(x-25)$, &c.

31. $2x(x+1)(x+5)+(x+3)(x-1)(x+5)=(3x+19)(x-1)(x+1)$,
i.e. $2x^3+12x^2+10x+x^3+7x^2+7x-15=3x^3+19x^2-3x-19$;

whence $20x=-4$; $\therefore\ x=-\frac{1}{5}$.

32. $4(x-2)=5(1-y)$; whence $4x+5y=13$.
Also $26x+3y=-4$;
$\therefore\ 12x+15y=39$ and $130x+15y=-20$; $\therefore\ 118x=-59$, &c.

33. $\frac{1}{(x-1)(x+1)}+\frac{1}{x-1}=\frac{7}{8}-\frac{1}{x+1}$;
$\therefore\ 8+8(x+1)=7(x-1)(x+1)-8(x-1)$;
whence $-7x^2+16x+15=0$; *i.e.* $(7x+5)(x-3)=0$, &c.

34. $\sqrt{5x+1}=2-\sqrt{x}$; $\therefore\ 5x+1=4-4\sqrt{x}+x$; whence $4x-3=-4\sqrt{x}$;
$\therefore\ 16x^2-24x+9=16x$; $\therefore\ 16x^2-40x+9=0$; *i.e.* $(4x-9)(4x-1)=0$, &c.

35. Let P and Q be the men, and let them ride at p and q miles an hour respectively. Let them meet at a place C, x hours after starting.

Now in x hours P travels px miles; $\therefore\ AC$ is px miles.

Similarly, since Q rides from C to A in 1 hour 20 minutes, AC is $\frac{4}{3}q$ miles.

Whence $px=\frac{4}{3}q$.......I.

In the same way, since Q rides from B to C in x hours, BC is qx miles.
And since P rides from C to B in 3 hours, BC is $3p$ miles.
Whence $qx=3p$.......II.

From I and II, $(px)(qx)=\left(\frac{4}{3}q\right)(3p)$; whence $x=2$.

36. $\frac{5x+8}{5x-1}-\frac{9}{(5x-1)(5x+1)}=\frac{10x+1}{5x+1}$;
$\therefore\ (5x+8)(5x+1)-9=(10x+1)(5x-1)$;
i.e. $25x^2+45x+8-9=50x^2-5x-1$;
whence $-25x^2+50x=0$; *i.e.* $25x(x-2)=0$, &c.

37. $5y-\frac{3}{x}+9=3y-\frac{2}{x}$; $\therefore\ 2y-\frac{1}{x}=-9$.

$5y-\frac{3}{x}+9=4y-\frac{7}{x}$; $\therefore\ y+\frac{4}{x}=-9$, or $2y+\frac{8}{x}=-18$.

Hence $\frac{9}{x}=-9$; $\therefore\ x=-1$.

Then $y=-9-\frac{4}{x}=-9+4=-5$.

38. Multiply by $4(3x-4)(x-7)$;

$$4(4x-7)(x-7)+4(2x+1)(3x-4)=73(3x-4)(x-7);$$
$$i.e.\ 16x^2-140x+196+24x^2-20x-16=219x^2-1825x+2044,$$
$$\therefore\ -179x^2+1665x-1864=0;\ \therefore\ (179x-233)(x-8)=0,\ \&c.$$

39. $\sqrt{4x+4}=\sqrt{x-4}+\sqrt{x+8}$;

$$\therefore\ 4x+4=(x-4)+2\sqrt{(x-4)(x+8)}+(x+8),\ \text{whence}\ 2x=2\sqrt{(x-4)(x+8)};$$
$$\therefore\ x^2=(x-4)(x+8);\ \text{whence}\ x=8.$$

40. Let x, y, and z shillings be their respective shares. Since A had one-ninth of the shares of B and C together, we have $x=\frac{1}{9}(y+z)$, whence,

$$9x-y-z=0\ldots\ldots\text{I.}$$

Similarly we have from the question,

$$y=\frac{1}{3}(z+x);\ \therefore\ -x+3y-z=0\ldots\ldots\text{II.}$$
$$z=x+y+6;\ \therefore\ -x-y+z=6\ldots\ldots\text{III.}$$

From I and III, $8x-2y=6$; from II and III, $-2x+2y=6$.

Hence $6x=12$; $\therefore\ x=2$, &c.

41. $\dfrac{3x-8}{4}-\dfrac{28-3x}{4}=5-\dfrac{15}{64}x$;

$$\therefore\ 16(3x-8)-16(28-3x)=320-15x;\ \text{whence}\ 111x=896.$$

42. $(x+3)(x-4)-(x-2)(x-3)=(x-3)(x-4)$;

$$i.e.\ x^2-x-12-x^2+5x-6=x^2-7x+12;$$
$$\therefore\ x^2-11x+30=0;\ i.e.\ (x-5)(x-6)=0,\ \&c.$$

43. Squaring, $x^2+4x+4=4+x\sqrt{8-x}$; $\therefore\ x^2+4x=x\sqrt{8-x}$.

Hence $x=0$; or else $x+4=\sqrt{8-x}$; $\therefore\ x^2+8x+16=8-x$;
whence $(x+1)(x+8)=0$. Thus $x=0$, or -1, or -8.

44. $3x=2y+12$; $\therefore\ (2y+12)^2-4y^2=576$;

$$\therefore\ 48y+144=576;\ \therefore\ y=9,\ \&c.$$

45. Let £100 stock cost £x. Then interest on £x is £$2\frac{3}{4}$; hence interest on £100 is $\dfrac{2\frac{3}{4}}{x}\times 100$.

At $x-7\frac{1}{3}$, the interest on £100 would be $\dfrac{2\frac{3}{4}}{x-7\frac{1}{3}}\times 100$.

Hence $\dfrac{2\frac{3}{4}}{x-7\frac{1}{3}}\times 100=\dfrac{2\frac{3}{4}}{x}\times 100+\dfrac{2}{9}$; $\therefore\ \dfrac{3300}{12x-88}=\dfrac{1100}{4x}+\dfrac{2}{9}$;
whence $3x^2-22x-27225=0$; $i.e.\ (x-99)(3x+275)=0$.

46. $(4x^3+4x^2+8x+1)(x+1)=(2x^2+2x+1)(2x^2+2x+3)$; multiplying out, we have,

$$4x^4+8x^3+12x^2+9x+1=4x^4+8x^3+12x^2+8x+3; \text{ whence } x=2.$$

47. $(\mathrm{I}\times 210)$; $21(5x+6)-10(11y-5)=2310$;

whence $105x-110y=2134$......III.

$(\mathrm{II}\times 25)$; $55y-12=35x-925$; $\therefore\ -35x+55y=-913$.......IV.

$(\mathrm{III}+\mathrm{IV}\times 2)$; $35x=308$; whence $x=\frac{44}{5}$.

$(\mathrm{III}+\mathrm{IV}\times 3)$; $55y=-605$; whence $y=-11$.

48. $36(x-1)+30x=19x(x-1)$;

whence $-19x^2+85x-36=0$; *i.e.* $(19x-9)(x-4)=0$, &c.

49. $\sqrt{x+4}=\frac{2}{\sqrt{x}}-\sqrt{x}$; $\therefore\ x+4=\frac{4}{x}-4+x$; whence $8=\frac{4}{x}$; $\therefore\ x=\frac{1}{2}$.

50. Let P and Q be the trains, and let them travel at p and q miles an hour respectively; let them meet at M, x hours after starting.

Then in x hours P travels px miles; but P travels from Cambridge to M in x hours; thus, the distance from Cambridge to M is px miles.

Similarly since Q travels from M to Cambridge in 1 hour 20 minutes $\left(i.e.\ \frac{4}{3}\text{ hours}\right)$, the distance from Cambridge to M is $\frac{4}{3}q$ miles.

Whence $px=\frac{4}{3}q$.......I.

In the same way since Q travels from London to M in x hours, the distance is qx miles.

And since P travels from M to London in $\frac{3}{4}$ hour, the distance is $\frac{3}{4}p$ miles.

Whence $qx=\frac{3}{4}p$.......II.

From I and II, $(px)(qx)=\left(\frac{4}{3}q\right)\left(\frac{3}{4}p\right)$; *i.e.* $pqx^2=pq$; whence $x=1$.

Thus P takes 1 hour from Cambridge to M and $\frac{3}{4}$ hour from M to London, *i.e.* $1\frac{3}{4}$ hours for the whole journey, &c.

51. $\frac{2x-1}{2(x-1)}-\frac{3}{5(x-1)}+\frac{1}{5}=\frac{23}{10(x-1)}$. Multiply by $10(x-1)$;

$$\therefore\ 5(2x-1)-6+2(x-1)=23; \text{ whence } x=3.$$

52. $5x+2y-1=3x-y+14$; whence $2x+3y=15$.

$3x-y+14=x+19y+6$; whence $2x-20y=-8$.

$$\therefore\ 23y=23;\ \therefore\ y=1; \text{ and then } x=6.$$

53. $\sqrt{(x^2-a^2)}+\dfrac{1}{\sqrt{(x^2-a^2)}}=2$; $\therefore$ squaring, $x^2-a^2+2+\dfrac{1}{x^2-a^2}=4$;

$$\therefore\ x^2-a^2-2+\frac{1}{x^2-a^2}=0;$$

$$\therefore\ (x^2-a^2)^2-2(x^2-a^2)+1=0;\ i.e.\ \{(x^2-a^2)-1\}^2=0;$$

$$\therefore\ (x^2-a^2)-1=0;\ \text{whence } x=\pm\sqrt{a^2+1}.$$

54. $7(3x^2-2xy+4y^2)=36(4x^2-y^2)$;

$$\therefore\ 123x^2+14xy-64y^2=0;\ i.e.\ (3x-2y)(41x+32y)=0.$$

If $3x-2y=0$, $4x^2-\left(\dfrac{3}{2}x\right)^2=7$; $\therefore$ $x^2=4$. Thus $x=\pm2$, and $y=\pm3$.

If $41x+32y=0$, $4x^2-\left(-\dfrac{41}{32}x\right)^2=7$; $\therefore$ $345x^2=1024$.

Thus $$x=\pm\frac{32}{\sqrt{345}},\ \text{and } y=\mp\frac{41}{\sqrt{345}}.$$

55. Let $x-1$, x, $x+1$ be the three numbers.
Then $(x+1)^3+(x-1)^3-2x^3=42$; whence $6x=42$; $\therefore$ $x=7$.

56. $$\frac{2\left(x-\frac{1}{2}\right)}{6}-\frac{3\left(\frac{x}{3}-12\right)}{15}=\frac{2x-1}{3}+\frac{2(9\frac{1}{2}-x)}{42};$$

$$i.e.\ \frac{2x-1}{6}-\frac{x-36}{15}=\frac{2x-1}{3}+\frac{19-2x}{42};$$

multiply by 210; $\therefore$ $35(2x-1)-14(x-36)=70(2x-1)+5(19-2x)$;
whence $$74x=444;\ \therefore\ x=6.$$

57. $$\frac{3(x-3)+2(x-2)}{6}=\frac{2(x-2)+3(x-3)}{(x-3)(x-2)};\ i.e.\ \frac{5x-13}{6}=\frac{5x-13}{x^2-5x+6};$$

thus either $5x-13=0$; whence $x=\dfrac{13}{5}$, or $x^2-5x+6=6$;
whence $$x(x-5)=0,\ \&c.$$

58. Multiply by a, b respectively, and add; then
$a^2x+b^2x=a^3-a^2b+ab^2-b^3$; $i.e.$ $x(a^2+b^2)=(a-b)(a^2+b^2)$; $\therefore$ $x=a-b$.
Then $a^2-ab-by=a^2-b^2-2ab$; whence $y=a+b$.

59. We have $$x^2+xy+y^2=9.$$
Also $(x^4+x^2y^2+y^4)\div(x^2+xy+y^2)=243\div9$, $i.e.$ $x^2-xy+y^2=27$.
Hence $$2xy=-18;\ \therefore\ xy=-9.$$
Hence $$x^2+2xy+y^2=0,\ i.e.\ (x+y)^2=0.$$
Also $$x^2-2xy+y^2=36;\ \therefore\ x-y=\pm6.$$
Whence $$x=\pm3\ \text{and}\ y=\mp3.$$

60. Suppose he bought x articles at y shillings each.

Then he sold $\frac{x}{2}$ for $\frac{x}{2} \times \frac{105}{100} y$; he sold $\frac{x}{3}$ for $\frac{x}{3} \times \frac{110}{100} y$; and $x - \frac{x}{2} - \frac{x}{3}$, that is $\frac{x}{6}$ for $\frac{x}{6} \times 2\frac{1}{4}$.

Also the total receipts were $x \times \frac{101\frac{2}{3}}{100} y$.

Hence
$$\frac{21}{40} xy + \frac{11}{30} xy + \frac{3}{8} x = \frac{61}{60} xy;$$

$$\therefore\ 63y + 44y + 45 = 122y; \text{ whence } y = 3.$$

61. $(a+b)\,x + (a+b)(a-b) + (a-b)\,x - (a-b)(a+b) + 2ax - 4ac = 0;$

$$\therefore\ 4ax - 4ac = 0; \quad \therefore\ x = c.$$

62. $3\,(3x-2)^2 - 3\,(2x-3)(2x-5) = 2\,(2x-5)(3x-2);$

i.e.
$$27x^2 - 36x + 12 - 12x^2 + 48x - 45 = 12x^2 - 38x + 20;$$

$$\therefore\ 3x^2 + 50x - 53 = 0; \text{ i.e. } (3x+53)(x-1) = 0, \text{ \&c.}$$

63. Put $x^2 + 2x + 14 = y^2$; then $x^2 + 2x = y^2 - 14$.

Then we have $2\,(x^2+2x) - 49 - 3\sqrt{(x^2+2x+14)} = 0;$

$$\therefore\ 2\,(y^2-14) - 49 - 3y = 0; \text{ whence } y = -\frac{11}{2}, \text{ or } 7.$$

(i) If $y = -\frac{11}{2}$; $x^2 + 2x + 14 = \frac{121}{4}$; whence $x = -1 \pm \frac{\sqrt{69}}{2}$.

(ii) If $y = 7$, $x^2 + 2x + 14 = 49$; whence $(x+7)(x-5) = 0$, &c.

64. $5\,(2x^2 + 3xy) = 2\,(y^2 - 2xy);$

whence $10x^2 + 19xy - 2y^2 = 0$; $(10x - y)(x + 2y) = 0$; $\therefore\ x = \frac{y}{10}$, or $x = -2y$.

(i) If $x = \frac{y}{10}$, $y^2 - \frac{y^2}{5} = 20$, whence $y = \pm 5$; $\therefore\ x = \frac{y}{10} = \pm\frac{1}{2}$.

(ii) If $x = -2y$, $y^2 + 4y^2 = 20$; whence $y = \pm 2$; $\therefore\ x = -2y = \mp 4$.

65. Suppose he bought x sheep, then each cost £$\frac{58}{x}$. He sold $\frac{1}{4}(x-5)$ at $\frac{105}{100}$ of £$\frac{58}{x}$, and received £$12\frac{3}{5}$.

$$\therefore\ \frac{1}{4}(x-5) \times \frac{105}{100} \times \frac{58}{x} = \frac{63}{5}; \quad \therefore\ 29\,(x-5) = 24x, \text{ whence } x = 29.$$

66.
$$\frac{1}{5x-1} + \frac{x+5}{(5x-1)(5x+1)} = \frac{3}{5x+1};$$

$$\therefore\ (5x+1) + (x+5) = 3\,(5x-1); \text{ whence } -9x = -9, \text{ \&c.}$$

67. $(a+b)x+by=ax+(a+b)y$; whence $bx-ay=0$.

Also $$ax+(a+b)y=a^3+b^3; \therefore ax+(a+b)\left(\frac{b}{a}x\right)=a^3+b^3;$$
$$\therefore (a^2+ab+b^2)x=a(a^3+b^3), \text{ \&c.}$$

68. $\sqrt{(x-a)}-\dfrac{b}{\sqrt{(x-a)}}=\dfrac{a}{\sqrt{(x-b)}}-\sqrt{(x-b)}$; thus, squaring,
$$(x-a)-2b+\frac{b^2}{x-a}=\frac{a^2}{x-b}-2a+(x-b);$$
whence $$a-b=\frac{a^2}{x-b}-\frac{b^2}{x-a};$$
$$\therefore (x-a)(x-b)(a-b)=x(a^2-b^2)-(a^3-b^3);$$
$$\therefore (x-b)(x-a)=x(a+b)-(a^2+ab+b^2);$$
whence $$x^2-2x(a+b)+(a+b)^2=0; \text{ i.e. } (x-a-b)^2=0, \text{ \&c.}$$

69. Subtract; then $2y^2=8$; whence $y=\pm 2$.

Then $$x^2+4=\pm 2x+7; \therefore x^2\mp 2x-3=0;$$
$$\therefore (x\mp 3)(x\pm 1)=0, \text{ \&c.}$$

70. Let x be the number of articles; then the original price of each was $\dfrac{100}{x}$ shillings. This price was raised to $\left(\dfrac{100}{x}+\dfrac{1}{2}\right)$ shillings, at which price $x-10$ articles would sell for $\left(\dfrac{100}{x}+\dfrac{1}{2}\right)(x-10)$ shillings.

Whence $$\left(\frac{100}{x}+\frac{1}{2}\right)(x-10)=100;$$
$$\therefore \frac{200+x}{2x}(x-10)=100; \therefore (200+x)(x-10)=200x;$$
$$\therefore x^2-10x-2000=0; \text{ i.e. } (x-50)(x+40)=0;$$
$\therefore x=50$ [the negative root being inapplicable].

71. Multiply by 30; then $6(3x+1)-10(2x-3)=3(6x-1)-5(7x-3)$;
$$\therefore 18x+6-20x+30=18x-3-35x+15; \therefore 15x=-24.$$

72. $\dfrac{x}{c}+\dfrac{a}{c}=\dfrac{c}{a}+\dfrac{c}{x}$;
$$\therefore \frac{x+a}{c}=\frac{c(x+a)}{ax}; \therefore ax(x+a)=c^2(x+a); \therefore (x+a)(ax-c^2)=0, \text{ \&c.}$$

73. $(x+1)+2\sqrt{2x}\sqrt{x+1}+2x=6x+1$; whence $2\sqrt{2x}\sqrt{x+1}=3x$;
$\therefore 4\{2x(x+1)\}=9x^2$; i.e. $8x^2+8x=9x^2$; whence $x(x-8)=0$, &c.

74. Add; then $x^2-2xy+y^2-3x+3y+1=11$;
$$\therefore (x-y)^2-3(x-y)-10=0; \text{ i.e. } \{(x-y)-5\}\{(x-y)+2\}=0;$$
whence $$x-y=5 \text{ or } -2.$$

(i) If $x-y=5$, $y=x-5$; and, from I,

$$x(x-y)+3y=11;\ \therefore\ 5x+3(x-5)=11;\ \therefore\ 8x=26;$$

whence $$x=\frac{13}{4},\text{ and } y=x-5=-\frac{7}{4}.$$

(ii) If $x-y=-2$, $y=x+2$, and, from I,

$$x(x-y)+3y=11;\ \therefore\ -2x+3(x+2)=11;\ \therefore\ x=5;\text{ whence } y=x+2=7.$$

75. Let there be x half-crowns, y shillings, and z sixpences. Since there are altogether 150 coins, $x+y+z=150$.......I.

The total value of the coins is 216 shillings.

Hence $$\frac{5}{2}x+y+\frac{z}{2}=216\text{.......II.}$$

The value of $2z$ sixpences, $\frac{x}{2}$ half-crowns, and $3y$ shillings is 311 shillings.

Hence $$\frac{5x}{4}+3y+z=311\text{.......III.}$$

Multiply the second equation by 2; then $5x+2y+z=432$.

Subtract from this the first, and also the third; then

$$4x+y=282,\text{ and } 3\tfrac{3}{4}x-y=121.$$

Hence $$7\tfrac{3}{4}x=403,\text{ \&c.}$$

76. Multiply by 105; then

$$21(3x+1)-35(1-5x)=35(x+1)-15(2-9x);$$

$$\therefore\ 63x+21-35+175x=35x+35-30+135x;\ \therefore\ 68x=19,\text{ \&c.}$$

77. Put $\frac{x^2+1}{x+1}=y$; then $\frac{1}{y}+y=\frac{29}{10}$;

whence $10y^2-29y+10=0$; *i.e.* $(2y-5)(5y-2)=0$; $\therefore\ y=\frac{5}{2}$, or $\frac{2}{5}$.

(i) If $y=\frac{5}{2}$, $\frac{x^2+1}{x+1}=\frac{5}{2}$;

whence $$2x^2-5x-3=0;\ i.e.\ (x-3)(2x+1)=0,\text{ \&c.}$$

(ii) If $y=\frac{2}{5}$; $\frac{x^2+1}{x+1}=\frac{2}{5}$;

whence $5x^2-2x+3=0$; whence, (§ 144),

$$x=\frac{2}{10}\pm\sqrt{\left(\frac{4}{100}-\frac{3}{5}\right)}=\frac{1}{5}\pm\sqrt{\left(-\frac{14}{25}\right)}.$$

78. Multiply the first equation by b; then $(bx-ab)(y-b)=ab^2$; thus, from II, $$(ay-ab)(y-b)=ab^2;\ \therefore\ a\{(y-b)^2-b^2\}=0;$$

$$i.e.\ ay(y-2b)=0,\ \therefore\ y=0,\text{ or } 2b.$$

From II, if $y=0$, $x=0$; and if $y=2b$, $x=2a$.

79. (I – II); $x^2-y^2=2(y-x)$; $\therefore\ x^2-y^2+2(x-y)=0$;
i.e. $(x-y)\{(x+y)+2\}=0$; whence $x=y$, or $x=-y-2$.

If $\quad x=y$, $y^2=2y+24$; whence $(y-6)(y+4)=0$;
$\therefore\ y=6$, or -4; and $x=y=6$, or -4.

If $\quad x=-y-2$; $y^2=-2y-4+24$;
$\therefore\ y^2+2y-20=0$; whence, (§ 144),

$$y=-\frac{2}{2}\pm\sqrt{\left(\frac{4}{4}+\frac{20}{1}\right)}=-1\pm\sqrt{(21)};\ \therefore\ x=-y-2=-1\mp\sqrt{(21)}.$$

80. Let the length be x feet, the breadth $x-8$ feet, and the height y feet.

Then the area is $2xy+2(x-8)y$; $\therefore\ 2xy+2(x-8)y=1120$. I.

If the height were $y+4$, the area of two ends and one side would be $x(y+4)+2(x-8)(y+4)$.

Hence $\quad x(y+4)+2(x-8)(y+4)=1120$.

These equations reduce to $xy-4y=280$ and $3xy+12x-16y=1184$.

From the first $\quad x=\dfrac{4y+280}{y}$;

$$\therefore\ 3(4y+280)+12\,\frac{4y+280}{y}-16y=1184;\quad \therefore\ \frac{3360}{y}-4y=296;$$

$\therefore\ y^2+74y-840=0$, *i.e.* $(y-10)(y+84)=0$. Hence $y=10$, &c.

81. Multiply by 18; then $9(x-2)-2(x-5)+15(x-1)=0$, &c.

82. $(a-x)^3+(b-x)^3=(a-x+b-x)^3$
$=(a-x)^3+(b-x)^3+3(a-x)^2(b-x)+3(a-x)(b-x)^2$.

Hence $\quad 3(a-x)^2(b-x)+3(a-x)(b-x)^2=0$,
i.e. $3(a-x)(b-x)(a-x+b-x)=0$, &c.

83. Since $x=y+1$, we have, from I,
$$(y-4)^2+(y-6)^2=2\{(y+1)y-40\};$$
$$\therefore\ y^2-8y+16+y^2-12y+36=2y^2+2y-80;$$
whence $-22y=-132$; $\therefore\ y=6$; then $x=y+1=7$.

84. $\quad x^2y^2+4xy-96=0$; *i.e.* $(xy+12)(xy-8)=0$;
thus $xy=-12$, or $xy=8$.

If $xy=-12$, $y=-\dfrac{12}{x}$; $\therefore\ x-\dfrac{12}{x}=6$; $\therefore\ x^2-6x-12=0$;

whence $x=3\pm\sqrt{21}$. Then $y=6-x=3\mp\sqrt{21}$.

If $xy=8$; $y=\dfrac{8}{x}$; $\therefore\ x+\dfrac{8}{x}=6$; whence $x^2-6x+8=0$;

i.e. $(x-2)(x-4)=0$; $\therefore\ x=2$ or 4; $\therefore\ y=6-x=4$ or 2.

85. Suppose he bought x tons and gained £y on the sale of them. Then from the question, y shillings was the cost price of a ton, and y pence the retail price of a cwt.

Since y shillings was the cost price of a ton, for x tons he gave xy shillings, *i.e.* £ $\frac{xy}{20}$, and since he gained £y on the sale, he sold the x tons for £ $\left(\frac{xy}{20}+y\right)$.

But, he sold the coal at the rate of y pence per cwt., *i.e.* £ $\frac{y}{12}$ per ton, thus he sold the x tons for £ $\frac{xy}{12}$.

Whence $\frac{xy}{20}+y=\frac{xy}{12}$; multiply by $\frac{60}{y}$; $\therefore\ 3x+60=5x$; whence $x=30$.

86. $$\frac{(2x-3)(2x-5)-(2x-4)^2}{(2x-4)(2x-5)}=\frac{(2x-6)(2x-8)-(2x-7)^2}{(2x-7)(2x-8)};$$
$$i.e.\ \frac{-1}{4x^2-18x+20}=\frac{-1}{4x^2-30x+56};$$
$$\therefore\ 4x^2-18x+20=4x^2-30x+56;\text{ whence } 12x=36, \&c.$$

87. $$2x(a-b)-(2x)^2+(a-1)(b-1)=0,$$
$$\therefore\ (2x)^2-2x\{(a-1)-(b-1)\}-(a-1)(b-1)=0;$$
$$i.e.\ \{2x-(a-1)\}\{2x+(b-1)\}=0;$$
whence $$x=\frac{a-1}{2},\text{ or } x=\frac{1-b}{2}.$$

88. $(6x+1)+2\sqrt{2(1-x)(6x+1)}+2(1-x)=7x+6$;

$\therefore\ 2\sqrt{2(1-x)(6x+1)}=3x+3$; $\therefore\ 4\{2(1-x)(6x+1)\}=9x^2+18x+9$;

$\therefore\ -48x^2+40x+8=9x^2+18x+9$; whence $(19x-1)(3x-1)=0$, &c.

89. Multiply the first equation by 3 and add the second; then
$x^3+y^3+3xy(x+y)=-1$, *i.e.* $(x+y)^3=-1$; $\therefore\ x+y=-1$.

Then $xy(x+y)=30$ gives $xy=-30$.

Hence $x-\frac{30}{x}=-1$; $\therefore\ x^2+x-30=0$, *i.e.* $(x+6)(x-5)=0$.

Hence $x=5$ or -6, and $\therefore\ y=-6$ or 5.

90. Let x and y yards respectively be lengths of the sides; then
$$xy=2420,\text{ and } 2x+2y=198.$$
Whence $x(99-x)=2420$; $\therefore\ x=44$ or 55; and then $y=55$ or 44.

91. $$(x+3)(2x-2)(x-2)+(3x+7)(2x+1)(x-2)=(2x+1)^2(2x-2);$$
$$\therefore\ 2x^3-14x+12+6x^3+5x^2-27x-14=8x^3-6x-2;$$
whence $$5x^2-35x=0;\ i.e.\ 5x(x-7)=0, \&c.$$

92. (II – I), ; $y+2-3z=4z-4$; whence $y-7z=-6$.
(I – III); $2y+z-3=4-4y$; whence $6y+z=7$.

$$\therefore\ 43y=43;\ \therefore\ y=1,\ \&c.$$

93. Take the cube of both sides;

$$(a-x)+3\{\sqrt[3]{(a-x)}\}^2\sqrt[3]{(x-b)}+3\sqrt[3]{(a-x)}\{\sqrt[3]{(x-b)}\}^2+(x-b)=a-b;$$

$$\therefore\ 3\{\sqrt[3]{(a-x)}\}^2\sqrt[3]{(x-b)}+3\sqrt[3]{(a-x)}\{\sqrt[3]{(x-b)}\}^2=0;$$

$$\therefore\ 3\sqrt[3]{(a-x)}\sqrt[3]{(x-b)}\{\sqrt[3]{(a-x)}+\sqrt[3]{(x-b)}\}=0;$$

whence and from I, $\sqrt[3]{(a-x)}\sqrt[3]{(x-b)}\{\sqrt[3]{(a-b)}\}=0$. Hence $x=a$ or b.

94. The second equation is $(x-3)(y-3)=15$. Put $x-3=a$, $y-3=b$; then we have $a^2+b^2=34$ and $ab=15$.

Hence $a^2+2ab+b^2=64$, and $a^2-2ab+b^2=4$.

$$\therefore\ a+b=\pm 8 \text{ and } a-b=\pm 2.$$

Whence $a=\pm 5$ or ± 3 and $b=\pm 3$ or ± 5.

That is $x-3=\pm 5$ or ± 3, and $y-3=\pm 3$ or ± 5.

Hence $x=8,\ -2,\ 6$ or 0, and $y=6,\ 0,\ 8$ or -2.

95. Let the rates of the trains be x and y miles per hour.

Then $(x+y)\times 1\tfrac{1}{4}=100$; also $\dfrac{100}{x}-\dfrac{100}{y}=1\tfrac{1}{3}$.

Thus $x+y=80$ and $\dfrac{1}{x}-\dfrac{1}{y}=\dfrac{1}{75}$.

Hence $\dfrac{1}{x}-\dfrac{1}{80-x}=\dfrac{1}{75}$;

$$\therefore\ x^2-230x+6000,\ i.e.\ (x-30)(x-200)=0.$$

Hence $x=30$ and $y=50$. [The values $x=200$, $y=-120$ being inapplicable.]

96. $7(ax-by)=3\{(2a+b)x-(a+2b)y\}$; whence $x(a-3b)=y(b-3a)$.

Hence, from $ax-by=3(a^2-b^2)$, $ax-bx\dfrac{a-3b}{b-3a}=3(a^2-b^2)$;

$\therefore\ 3x(b^2-a^2)=3(a^2-b^2)(b-3a)$; $\therefore\ x=3a-b$, and thus $y=3b-a$.

97. $13x^2-80x+12=0$, *i.e.* $(13x-2)(x-6)=0$, &c.

98. $(x+1)(x+5)=x^2+6x+5$, and $(x+2)(x+4)=x^2+6x+8$.

Hence $(x^2+6x+5)(x^2+6x+8)=4$;

$\therefore\ (x^2+6x)^2+13(x^2+6x)+36=0$, *i.e.* $(x^2+6x+4)(x^2+6x+9)=0$, &c.

99. We have $x^2y+xy^2+x^2+y^2=109$;

$$\therefore\ xy(x+y)+x^2+2xy+y^2=109+2xy.$$

But $xy=12$; hence $12(x+y)+(x+y)^2=109+24$;

$\therefore\ (x+y)^2+12(x+y)-133=0$, whence $x+y=7$ or $x+y=-19$.

From $x+y=7$ and $xy=12$, we have $x=3$, $y=4$, or $x=4$, $y=3$.
From $x+y=-19$ and $xy=12$, we have

$$x=-\frac{19}{2}\pm\frac{1}{2}\sqrt{313},\quad y=-\frac{19}{2}\mp\frac{1}{2}\sqrt{313}.$$

100. Let the distance from P to Q be d miles, and suppose A and B to walk at the rates of x and y miles per hour.

Then A walked $d+\frac{d}{2}$ in 6 hours; $\therefore\ \frac{3}{2}d=6x$, or $x=\frac{1}{4}d$.

B walked $d+\frac{d}{2}$ in $7\frac{1}{2}$ hours; $\therefore\ \frac{3}{2}d=7\frac{1}{2}y$, or $y=\frac{1}{5}d$.

Also it took A 24 minutes longer to walk $d-6$ miles than it took B to walk 6 miles. Hence $\frac{d-6}{x}=\frac{6}{y}+\frac{24}{60}$.

Hence $$\frac{4d-24}{d}=\frac{30}{d}+\frac{2}{5};\quad\therefore\ \frac{54}{d}=\frac{18}{5};\quad\therefore\ d=15.$$

LIII. PAGE 214.

16. $(2a^4)^2+2(2a^4)(3b^3)+(3b^3)^2=$&c.

17. $(a^5)^2+2(a^5)(-2b^4)+(-2b^4)^2=$&c.

18. $(-ax^4)^2+2(-ax^4)(by^3)+(by^3)^2=$&c.

19. $(-a^2)^2+2(-a^2)(2ab)+(2ab)^2=$&c.

20. $(a^2)^3+3(a^2)^2b^2+3a^2(b^2)^2+(b^2)^3=$&c.

21. $(a^3)^3+3(a^3)^2b^3+3a^3(b^3)^2+(b^3)^3=$&c.

22. $(2a^2)^3+3(2a^2)^2(-3b^2)+3(2a^2)(-3b^2)^2+(-3b^2)^3=$&c.

23. $(3a^2)^3+3(3a^2)^2(-2b^2)+3(3a^2)(-2b^2)^2+(-2b^2)^3=$&c.

24. $a^4+b^4+c^4+2a^2b^2+2a^2c^2+2b^2c^2$.

25. $a^6+4b^6+9c^6+2a^3(-2b^3)+2a^3(3c^3)+2(-2b^3)(3c^3)=$&c.

26. $a^4+16b^4+9c^4+2a^2(-4b^2)+2a^2(-3c^2)+2(-4b^2)(-3c^2)=$&c.

27. $x^4+9x^2+36+2x^2(-3x)+2x^2(-6)+2(-3x)(-6)=$&c.

28. $9x^4+x^2+25+2(3x^2)(-x)+2(3x^2)(-5)+2(-x)(-5)=$&c.

29. $4x^4+25x^2+1+2(2x^2)(5x)+2(2x^2)(-1)+2(5x)(-1)=$&c.

30. $9x^4+36x^2+36+2(3x^2)(-6x)+2(3x^2)(-6)+2(-6x)(-6)=$&c.

31. $1+x^2+x^4+x^6+2x+2x^2+2x^3+2x\,.\,x^2+2x\,.\,x^3+2x^2\,.\,x^3=$&c.

32. $x^6+x^4+x^2+1+2x^3(-x^2)+2x^3\,.\,x+2x^3(-1)+2(-x^2)x$
$+2(-x^2)(-1)+2x(-1)=$&c.

33. $x^6+x^4+4x^2+4+2x^3\,.\,x^2+2x^3(-2x)+2x^3(-2)+2x^2(-2x)$
$+2x^2(-2)+2(-2x)(-2)=$&c.

34. $a^2+4b^2+9c^2+16d^2+2a\,.\,2b+2a\,.\,3c+2a\,.\,4d+2\,.\,2b\,.\,3c$
$+2\,.\,2b\,.\,4d+2\,.\,3c\,.\,4d=$&c.

35. $4a^2+b^2+c^2+4d^2+2\,.\,2a\,(-b)+2\,.\,2a\,.\,c+2\,.\,2a\,(-2d)$
$+2\,(-b)\,c+2\,(-b)\,(-2d)+2c\,(-2d)=\&c.$

36. $(x^2)^3+3\,(x^2)^2\,(x+1)+3x^2\,(x+1)^2+(x+1)^3=\&c.$

37. $(x^2)^3+3\,(x^2)^2\,(-x+2)+3x^2\,(-x+2)^2+(-x+2)^3=\&c.$

38. $(3x^2)^3+3\,(3x^2)^2\,(-5x+1)+3\,(3x^2)\,(-5x+1)^2+(-5x+1)^3=\&c.$

LIV. PAGE 219.

13. $\sqrt{\{a^2+2a\,(2b+3c)+(4b^2+12bc+9c^2)\}}=a+2b+3c.$

14. $\sqrt{\{4a^2-4a\,(b+3c)+(b^2+6bc+9c^2)\}}=2a-(b+3c).$

15. $\sqrt{\{4a^4+4a^2\,(b^2-c^2)+(b^4-2b^2c^2+c^4)\}}=2a^2+b^2-c^2.$

16. $\sqrt{\{25a^4+2\,.\,5a^2\,(-3b^2+2c^2)+(9b^4-12b^2c^2+4c^4)\}}=5a^2-3b^2+2c^2.$

LV. PAGE 222.

1.

$$\begin{array}{ll} & x^4+2x^3+3x^2+2x+1\ (\ x^2+x+1 \\ (x^2)^2 & =x^4 \\ (x^2+x)^2 & =x^4+2x^3+\ \ x^2 \\ (x^2+x+1)^2 & =x^4+2x^3+3x^2+2x+1 \end{array}$$

2.

$$\begin{array}{ll} & 4x^4-8x^3+4x+1\ (\ 2x^2-2x-1 \\ (2x^2)^2 & =4x^4 \\ (2x^2-2x)^2 & =4x^4-8x^3+4x^2 \\ (2x^2-2x-1)^2 & =4x^4-8x^3\qquad +4x+1 \end{array}$$

3.

$$\begin{array}{ll} & 9x^4-36x^3+72x+36\ (\ 3x^2-6x-6 \\ (3x^2)^2 & =9x^4 \\ (3x^2-6x)^2 & =9x^4-36x^3+36x^2 \\ (3x^2-6x-6)^2 & =9x^4-36x^3\qquad +72x+36 \end{array}$$

4.

$$\begin{array}{ll} & 1-xy-\dfrac{15}{4}x^2y^2+2x^3y^3+4x^4y^4\ \Big(\ 1-\dfrac{1}{2}xy-2x^2y^2 \\ 1^2 & =1 \\ \left(1-\dfrac{1}{2}xy\right)^2 & =1-xy+\dfrac{1}{4}x^2y^2 \\ \left(1-\dfrac{1}{2}xy-2x^2y^2\right)^2 & =1-xy-\dfrac{15}{4}x^2y^2+2x^3y^3+4x^4y^4 \end{array}$$

5.

$$\begin{array}{ll} & 4x^4+4x^3-\dfrac{1}{2}x+\dfrac{1}{16}\ \Big(\ 2x^2+x-\dfrac{1}{4} \\ (2x^2)^2 & =4x^4 \\ (2x^2+x)^2 & =4x^4+4x^3+x^2 \\ \left(2x^2+x-\dfrac{1}{4}\right)^2 & =4x^4+4x^3\qquad -\dfrac{1}{2}x+\dfrac{1}{16} \end{array}$$

6.
$$\begin{array}{ll}
 & x^4 - 2x^3 + \frac{3}{2}x^2 - \frac{1}{2}x + \frac{1}{16} \left(x^2 - x + \frac{1}{4} \right. \\
(x^2)^2 & = x^4 \\
(x^2 - x)^2 & = x^4 - 2x^3 + \ x^2 \\
\left(x^2 - x + \frac{1}{4} \right)^2 & = x^4 - 2x^3 + \frac{3}{2}x^2 - \frac{1}{2}x + \frac{1}{16}
\end{array}$$

7.
$$\begin{array}{ll}
 & x^4 + 2x^3y + 3x^2y^2 + 2xy^3 + y^4 \ (\ x^2 + xy + y^2 \\
(x^2)^2 & = x^4 \\
(x^2 + xy)^2 & = x^4 + 2x^3y + x^2y^2 \\
(x^2 + xy + y^2)^2 & = x^4 + 2x^3y + 3x^2y^2 + 2xy^3 + y^4
\end{array}$$

8.
$$\begin{array}{ll}
 & 16 - 96x + 216x^2 - 216x^3 + 81x^4 \ (\ 4 - 12x + 9x^2 \\
4^2 & = 16 \\
(4 - 12x)^2 & = 16 - 96x + 144x^2 \\
(4 - 12x + 9x^2)^2 & = 16 - 96x + 216x^2 - 216x^3 + 81x^4
\end{array}$$

9.
$$\begin{array}{ll}
 & 1 + 4x + 10x^2 + 12x^3 + 9x^4 \ (\ 1 + 2x + 3x^2 \\
1^2 & = 1 \\
(1 + 2x)^2 & = 1 + 4x + \ 4x^2 \\
(1 + 2x + 3x^2)^2 & = 1 + 4x + 10x^2 + 12x^3 + 9x^4
\end{array}$$

10.
$$\begin{array}{ll}
 & 4x^4 - 4x^3 + 3x^2 - x + \frac{1}{4} \left(\ 2x^2 - x + \frac{1}{2} \right. \\
(2x^2)^2 & = 4x^4 \\
(2x^2 - x)^2 & = 4x^4 - 4x^3 + \ x^2 \\
\left(2x^2 - x + \frac{1}{2} \right)^2 & = 4x^4 - 4x^3 + 3x^2 - x + \frac{1}{4}
\end{array}$$

11. The expression is $4x^4 + 8x^3 - 4x + 1$.
$$\begin{array}{ll}
 & 4x^4 + 8x^3 - 4x + 1 \ (\ 2x^2 + 2x - 1 \\
(2x^2)^2 & = 4x^4 \\
(2x^2 + 2x)^2 & = 4x^4 + 8x^3 + 4x^2 \\
(2x^2 + 2x - 1)^2 & = 4x^4 + 8x^3 - 4x + 1
\end{array}$$

12.
$$\begin{array}{ll}
 & x^6 - 4x^5 + 6x^4 - 8x^3 + 9x^2 - 4x + 4 \ (\ x^3 - 2x^2 + x - 2 \\
(x^3)^2 & = x^6 \\
(x^3 - 2x^2)^2 & = x^6 - 4x^5 + 4x^4 \\
(x^3 - 2x^2 + x)^2 & = x^6 - 4x^5 + 6x^4 - 4x^3 + x^2 \\
(x^3 - 2x^2 + x - 2)^2 & = x^6 - 4x^5 + 6x^4 - 8x^3 + 9x^2 - 4x + 4
\end{array}$$

We may also use the method of Art. 177. Thus

13.
$$\begin{array}{rl}
 & 9x^6 - 12x^5 + 22x^4 + x^2 + 12x + 4 \ (\ 3x^3 - 2x^2 + 3x + 2 \\
 & 9x^6 \\
6x^3 - 2x^2 \ ) & -12x^5 \\
 & -12x^5 + \ 4x^4 \\
6x^3 - 4x^2 + 3x \ ) & 18x^4 \\
 & 18x^4 - 12x^3 + 9x^2 \\
6x^3 - 4x^2 + 6x + 2 \ ) & 12x^3 - 8x^2 + 12x + 4 \\
 & 12x^3 - 8x^2 + 12x + 4
\end{array}$$

14.

$$\begin{array}{rl}
 & x^6-22x^4+34x^3+121x^2-374x+289\ (\,x^3-11x+17 \\
 & x^6 \\
2x^3-11x\ ) & \overline{-22x^4} \\
 & -22x^4 \qquad\quad +121x^2 \\
2x^3-22x+17\ ) & \overline{34x^3} \\
 & 34x^3 \qquad\quad -374x+289
\end{array}$$

15. $a^2-2a\left(\frac{1}{2}x-4\right)+\left(\frac{1}{4}x^2-4x+16\right)=\{a-\tfrac{1}{2}(x-4)\}^2.$

16.

$$\begin{array}{rl}
 & x^8+2x^7+x^6-4x^5-12x^4-8x^3+4x^2+16x+16\ (\,x^4+x^3-2x-4 \\
 & x^8 \\
2x^4+x^3\ ) & \overline{2x^7} \\
 & 2x^7+x^6 \\
2x^4+2x^3-2x\ ) & \overline{-4x^5} \\
 & -4x^5-\ 4x^4 \qquad +4x^2 \\
2x^4+2x^3-4x-4\ ) & \overline{-\ 8x^4-8x^3} \\
 & -\ 8x^4-8x^3 \qquad +16x+16
\end{array}$$

17.

$$\begin{array}{rl}
 & 16x^2-96x+216-\frac{216}{x}+\frac{81}{x^2}\left(4x-12+\frac{9}{x}\right. \\
 & 16x^2 \\
8x-12\ ) & \overline{-96x} \\
 & -96x+144 \\
8x-24+\frac{9}{x}\Big) & \overline{72} \\
 & 72-\frac{216}{x}+\frac{81}{x^2}
\end{array}$$

18.

$$\begin{array}{rl}
 & x^6-6x^4+15x^2-20+\frac{15}{x^2}-\frac{6}{x^4}+\frac{1}{x^6}\left(x^3-3x+\frac{3}{x}-\frac{1}{x^3}\right. \\
 & x^6 \\
2x^3-3x\ ) & \overline{-6x^4} \\
 & -6x^4+\ 9x^2 \\
2x^3-6x+\frac{3}{x}\Big) & \overline{6x^2} \\
 & 6x^2-18+\frac{9}{x^2} \\
2x^3-6x+\frac{6}{x}-\frac{1}{x^3}\Big) & \overline{-\ 2+\frac{6}{x^2}} \\
 & -\ 2+\frac{6}{x^2}-\frac{6}{x^4}+\frac{1}{x^6}
\end{array}$$

19.

$$\begin{array}{rl}
 & x^4+2x^3(y+z)+x^2(y^2+z^2+4yz)+2xyz(y+z)+y^2z^2\ \big(\ x^2+x(y+z)+ \\
 & \underline{x^4\qquad\qquad\qquad} \\
2x^2+x(y+z)\ \big) & 2x^3(y+z) \\
 & \underline{2x^3(y+z)+x^2(y^2+z^2+2yz)} \\
2x^2+2x(y+z)+yz\ \big) & x^2(2yz) \\
 & x^2(2yz)\qquad\qquad +2xyz(y+z)+y^2z^2
\end{array}$$

20. $2x^2(y^2+2yz+z^2)+2y^2(z^2+2xz+x^2)+2z^2(x^2+2xy+y^2)+4xyz(x+y+z)$
$=x^2(4y^2+8yz+4z^2)+2x(4y^2z+4yz^2)+4y^2z^2$
$=\{x(2y+2z)\}^2+2x(2y+2z)(2yz)+(2yz)^2;$
whence square root is $x(2y+2z)+2yz$.

LVI. Page 228.

1. $\{\sqrt[3]{8}\}^2=2^2=4.$ 2. $\dfrac{1}{4^{\frac{3}{2}}}=\dfrac{1}{(\sqrt{4})^3}=\dfrac{1}{2^3}.$

3. $\dfrac{1}{16^{\frac{1}{2}}}=\dfrac{1}{\sqrt{16}}=\dfrac{1}{4}.$ 4. $\dfrac{1}{25^{-\frac{1}{2}}}=25^{\frac{1}{2}}=\sqrt{25}=5.$

5. $\dfrac{4^{-\frac{3}{2}}}{25^{-\frac{3}{2}}}=\dfrac{25^{\frac{3}{2}}}{4^{\frac{3}{2}}}=\dfrac{(\sqrt{25})^3}{(\sqrt{4})^3}=\dfrac{5^3}{2^3}=\dfrac{125}{8}.$ 6. $\dfrac{27^{-\frac{2}{3}}}{64^{-\frac{2}{3}}}=\dfrac{64^{\frac{2}{3}}}{27^{\frac{2}{3}}}=\dfrac{(\sqrt[3]{64})^2}{(\sqrt[3]{27})^2}=\dfrac{4^2}{3^2}=\dfrac{16}{9}.$

7. $\dfrac{1}{27^{\frac{4}{3}}}=\dfrac{1}{(\sqrt[3]{27})^4}=\dfrac{1}{3^4}=\dfrac{1}{81}.$ 8. $\dfrac{1}{100^{\frac{5}{2}}}=\dfrac{1}{(\sqrt{100})^5}=\dfrac{1}{10^5}=\dfrac{1}{100000}.$

9. $\dfrac{1}{(10000)^{-\frac{3}{4}}}=10000^{\frac{3}{4}}=(\sqrt[4]{10000})^3=10^3=1000.$

10. $\dfrac{27^{-\frac{4}{3}}}{125^{-\frac{4}{3}}}=\dfrac{125^{\frac{4}{3}}}{27^{\frac{4}{3}}}=\dfrac{(\sqrt[3]{125})^4}{\sqrt[3]{(27)^4}}=\dfrac{5^4}{3^4}=\dfrac{625}{81}.$

11. $a^{\frac{2}{3}-\frac{1}{3}}=a^{\frac{1}{3}}.$ 12. $a^{\frac{4}{3}+\frac{2}{3}}=a^2.$

13. $a^{-\frac{2}{3}+1}=a^{\frac{1}{3}}.$ 14. $a^{1-\frac{2}{5}}=a^{\frac{3}{5}}.$

15. $a^{\frac{2}{3}-\frac{1}{3}}\,b^{-\frac{1}{3}+\frac{2}{3}}=a^{\frac{1}{3}}\,b^{\frac{1}{3}}.$ 16. $a^{\frac{1}{2}\times 2}\,b^{1\times 2}=ab^2.$

17. $a^{\frac{3}{2}\times 2}\,b^{-\frac{1}{2}\times 2}=a^3b^{-1}.$ 18. $a^{\frac{1}{3}\times 3}\,b^{-\frac{2}{3}\times 3}=ab^{-2}.$

19. $a^{-\frac{2}{3}\times\frac{3}{2}}\,b^{\frac{1}{6}\times\frac{3}{2}}=a^{-1}b^{\frac{1}{4}}.$ 20. $a^{4\times\frac{3}{2}}\,b^{-\frac{2}{3}\times\frac{3}{2}}=a^6b^{-1}.$

21. $\{a^{-2\times 2}\}^{\frac{3}{4}}=a^{-2\times 2\times\frac{3}{4}}=a^{-3}.$ 22. $\{a^{-\frac{1}{3}\times 2}\}^{-\frac{3}{2}}=a^{-\frac{1}{3}\times 2\times\left(-\frac{3}{2}\right)}=a.$

23. $a^{-\frac{5}{6}\times 3\times\left(-\frac{1}{5}\right)}=a^{\frac{1}{2}}.$ 24. $a^{\frac{1}{3}-\frac{3}{4}+\frac{5}{12}}=a^0=1.$

25. $a^{\frac{2}{3}} \times a^{-\frac{3}{4}} \times a^{2\times\left(-\frac{1}{6}\right)} \times a^{\frac{1}{12}\times 5} = a^{\frac{2}{3}} \times a^{-\frac{3}{4}} \times a^{-\frac{1}{3}} \times a^{\frac{5}{12}} = a^{\frac{2}{3}-\frac{3}{4}-\frac{1}{3}+\frac{5}{12}} = a^0 = 1.$

26. $x^{p+q} \times x^{p-q} \times \frac{1}{x^{2p}} = x^{p+q} \times x^{p-q} \times x^{-2p} = x^{p+q+p-q-2p} = x^0 = 1.$

27. $x^{(q-r)p} \times x^{(r-p)q} \times x^{(p-q)r} = x^{(q-r)p+(r-p)q+(p-q)r} = x^0 = 1.$

28. $\frac{x^{\frac{1}{2}}y^{\frac{2}{3}}}{x^{\frac{1}{3}}y^{\frac{1}{2}}} \times \frac{y^{\frac{1}{6}}}{x^{\frac{1}{6}}} = x^{\frac{1}{2}}y^{\frac{2}{3}}x^{-\frac{1}{3}}y^{-\frac{1}{2}} \times y^{\frac{1}{6}}x^{-\frac{1}{6}} = x^{\frac{1}{2}-\frac{1}{3}-\frac{1}{6}}y^{\frac{2}{3}-\frac{1}{2}+\frac{1}{6}} = x^0y^{\frac{1}{3}} = y^{\frac{1}{3}}.$

29. $x^{x+y}y^{x+y}x^{-y}y^{-x} = x^{x+y-y}y^{x+y-x} = x^xy^y.$

30. $x^{x+y+z}y^{x+y+z}z^{x+y+z}x^{-(y+z)}y^{-(z+x)}z^{-(x+y)}$

$= x^{x+y+z-(y+z)}y^{x+y+z-(z+x)}z^{x+y+z-(x+y)} = x^xy^yz^z.$

31. $\frac{a^{\frac{1}{6}}b^3}{b^{\frac{3}{4}}a^{\frac{1}{3}}} \div \frac{a^{\frac{1}{6}}b^{\frac{1}{12}}}{b^{\frac{6}{7}}} = \frac{a^{\frac{1}{6}}b^3}{b^{\frac{3}{4}}a^{\frac{1}{3}}} \times \frac{b^{\frac{6}{7}}}{a^{\frac{1}{6}}b^{\frac{1}{12}}} = a^{\frac{1}{6}}b^3b^{-\frac{3}{4}}a^{-\frac{1}{3}} \times b^{\frac{6}{7}}a^{-\frac{1}{6}}b^{-\frac{1}{12}}$

$= a^{\frac{1}{6}-\frac{1}{3}-\frac{1}{6}}b^{3-\frac{3}{4}+\frac{6}{7}-\frac{1}{12}} = a^{-\frac{1}{3}}b^{\frac{127}{42}}.$

32. $a^{\frac{1}{2}}y^{\frac{1}{3}} \times \frac{y^{\frac{1}{2}}}{x^{\frac{1}{3}}} \times \frac{x^{\frac{1}{4}}}{y^{-\frac{1}{4}}} = a^{\frac{1}{2}}y^{\frac{1}{3}} \times y^{\frac{1}{2}}x^{-\frac{1}{3}} \times x^{\frac{1}{4}}y^{\frac{1}{4}} = a^{\frac{1}{2}}x^{-\frac{1}{3}+\frac{1}{4}}y^{\frac{1}{3}+\frac{1}{2}+\frac{1}{4}} = a^{\frac{1}{2}}x^{-\frac{1}{12}}y^{\frac{13}{12}}.$

33. $x^{3p+q} \times x^{p-4r} \times x^{2(q-2r)} \times x^{8r-4p} = x^{3p+q+p-4r+2q-4r+8r-4p} = \&c.$

34. $x^{\frac{(p-q)r}{pq}}x^{\frac{(q-r)p}{qr}}x^{\frac{(r-p)q}{rp}} \times \frac{1}{x^{\frac{p-q}{r}+\frac{q-r}{p}+\frac{r-p}{q}}}$

$= x^{\frac{(p-q)r}{pq}+\frac{(q-r)p}{qr}+\frac{(r-p)q}{rp}} \times x^{-\frac{p-q}{r}-\frac{q-r}{p}-\frac{r-p}{q}}$

$= x^{\frac{(p-q)r}{pq}+\frac{(q-r)p}{qr}+\frac{(r-p)q}{rp}-\frac{p-q}{r}-\frac{q-r}{p}-\frac{r-p}{q}}$

$= x^0 = 1.$

35. $x^{\frac{p-q}{r}\cdot\frac{q-r}{p}\cdot\frac{r-p}{q}} \times x^{\frac{p-q}{r}+\frac{q-r}{p}+\frac{r-p}{q}} = x^{\frac{p-q}{r}\cdot\frac{q-r}{p}\cdot\frac{r-p}{q}+\frac{p-q}{r}+\frac{q-r}{p}+\frac{r-p}{q}} = x^0 = 1.$

36. $a^{\frac{1}{3}} \times a^{\frac{1}{2}} \times a^{\frac{1}{6}} = a^{\frac{1}{3}+\frac{1}{2}+\frac{1}{6}} = a^1 = a.$

37. $(a^2y)^{\frac{1}{3}} \times (ay^2)^{\frac{1}{3}} = a^{\frac{2}{3}}y^{\frac{1}{3}} \times a^{\frac{1}{3}}y^{\frac{2}{3}} = a^{\frac{2}{3}+\frac{1}{3}}y^{\frac{1}{3}+\frac{2}{3}} = ay.$

38. $(a^2x^5)^{\frac{1}{3}} \times (a^3x)^{\frac{1}{2}} = a^{\frac{2}{3}}x^{\frac{5}{3}} \times a^{\frac{3}{2}}x^{\frac{1}{2}} = a^{\frac{2}{3}+\frac{3}{2}}x^{\frac{5}{3}+\frac{1}{2}} = \&c.$

39. $a^{\frac{3}{7}} \times a^{\frac{7}{6}} \times a^{-\frac{1}{2}} \times a^{-\frac{2}{21}} = a^{\frac{3}{7}+\frac{7}{6}-\frac{1}{2}-\frac{2}{21}} = a.$

40. $a^{\frac{7}{3}} \times a^{\frac{7}{5}} \times a^{-\frac{1}{3}} \times a^{-\frac{4}{5}} = a^{\frac{7}{3}+\frac{7}{5}-\frac{1}{3}-\frac{4}{5}} = \&c.$

41. $\sqrt[4]{a^2\sqrt[3]{\{a^6\times a^{-\frac{23}{2}}\}}}=\sqrt[4]{a^2\sqrt[3]{\{a^{-\frac{11}{2}}\}}}=\sqrt[4]{a^2\times a^{-\frac{11}{6}}}=\sqrt[4]{a^{\frac{1}{6}}}=a^{\frac{1}{24}}.$

42. $a^{\frac{2}{5}}b^2c\times\dfrac{1}{a^{\frac{1}{3}}b^2c}=a^{\frac{2}{5}}b^2c\times a^{-\frac{1}{3}}b^{-2}c^{-1}=a^{\frac{2}{5}-\frac{1}{3}}b^{2-2}c^{1-1}=a^{\frac{1}{15}}.$

43. $a^{\frac{3}{2}}b^{-1}\times\dfrac{1}{a^{-\frac{4}{3}}b^{\frac{5}{3}}}=a^{\frac{3}{2}}b^{-1}\times a^{\frac{4}{3}}b^{-\frac{5}{3}}=a^{\frac{3}{2}+\frac{4}{3}}b^{-1-\frac{5}{3}}=\&c.$

44. $a^2b^{\frac{1}{2}}c^{\frac{1}{6}}\times b^{-\frac{9}{2}}\times c^{-\frac{1}{6}}a^{-2}=a^{2-2}b^{\frac{1}{2}-\frac{9}{2}}c^{\frac{1}{6}-\frac{1}{6}}=b^{-4}.$

LVII. Pages 230—231.

1.
$$\begin{array}{lll}
x^{\frac{1}{2}}+y^{\frac{1}{2}} & & \\
x^{\frac{1}{2}}-y^{\frac{1}{2}} & & \\
\hline
x & +x^{\frac{1}{2}}y^{\frac{1}{2}} & \\
 & -x^{\frac{1}{2}}y^{\frac{1}{2}} & -y \\
\hline
x & & -y
\end{array}$$

2.
$$\begin{array}{lll}
x^{\frac{3}{2}}+y^{\frac{3}{2}} & & \\
x^{\frac{3}{2}}-y^{\frac{3}{2}} & & \\
\hline
x^3 & +x^{\frac{3}{2}}y^{\frac{3}{2}} & \\
 & -x^{\frac{3}{2}}y^{\frac{3}{2}} & -y^3 \\
\hline
x^3 & & -y^3
\end{array}$$

3.
$$\begin{array}{lll}
x^{\frac{1}{4}}+1 & & \\
x^{\frac{1}{4}}-1 & & \\
\hline
x^{\frac{1}{2}} & +x^{\frac{1}{4}} & \\
 & -x^{\frac{1}{4}} & -1 \\
\hline
x^{\frac{1}{2}} & & -1
\end{array}$$

4.
$$\begin{array}{lll}
x^{\frac{1}{3}}+y^{\frac{1}{3}} & & \\
x^{\frac{1}{3}}-y^{\frac{1}{3}} & & \\
\hline
x^{\frac{2}{3}} & +x^{\frac{1}{3}}y^{\frac{1}{3}} & \\
 & -x^{\frac{1}{3}}y^{\frac{1}{3}} & -y^{\frac{2}{3}} \\
\hline
x^{\frac{2}{3}} & & -y^{\frac{2}{3}}
\end{array}$$

5.
$$\begin{array}{llll}
x^{\frac{2}{3}}+x^{\frac{1}{3}}+1 & & & \\
x^{\frac{1}{3}}-1 & & & \\
\hline
x & +x^{\frac{2}{3}} & +x^{\frac{1}{3}} & \\
 & -x^{\frac{2}{3}} & -x^{\frac{1}{3}} & -1 \\
\hline
x & & & -1
\end{array}$$

6.
$$\begin{array}{llll}
x^{\frac{2}{3}}-x^{\frac{1}{3}}y^{\frac{1}{3}}+y^{\frac{2}{3}} & & & \\
x^{\frac{1}{3}}+y^{\frac{1}{3}} & & & \\
\hline
x & -x^{\frac{2}{3}}y^{\frac{1}{3}} & +x^{\frac{1}{3}}y^{\frac{2}{3}} & \\
 & +x^{\frac{2}{3}}y^{\frac{1}{3}} & -x^{\frac{1}{3}}y^{\frac{2}{3}} & +y \\
\hline
x & & & +y
\end{array}$$

7.
$$\begin{array}{llll}
a^{\frac{4}{3}}+a^{\frac{2}{3}}b^{\frac{2}{3}}+b^{\frac{4}{3}} & & & \\
a^{\frac{2}{3}}-b^{\frac{2}{3}} & & & \\
\hline
a^2 & +a^{\frac{4}{3}}b^{\frac{2}{3}} & +a^{\frac{2}{3}}b^{\frac{4}{3}} & \\
 & -a^{\frac{4}{3}}b^{\frac{2}{3}} & -a^{\frac{2}{3}}b^{\frac{4}{3}} & -b^2 \\
\hline
a^2 & & & -b^2
\end{array}$$

8.
$$\begin{array}{lllll}
a^{\frac{4}{3}}+a^{\frac{2}{3}}b^{\frac{2}{3}}+b^{\frac{4}{3}} & & & & \\
a^{\frac{4}{3}}-a^{\frac{2}{3}}b^{\frac{2}{3}}+b^{\frac{4}{3}} & & & & \\
\hline
a^{\frac{8}{3}} & +a^2b^{\frac{2}{3}} & +a^{\frac{4}{3}}b^{\frac{4}{3}} & & \\
 & -a^2b^{\frac{2}{3}} & -a^{\frac{4}{3}}b^{\frac{4}{3}} & -a^{\frac{2}{3}}b^2 & \\
 & & a^{\frac{4}{3}}b^{\frac{4}{3}} & +a^{\frac{2}{3}}b^2 & +b^{\frac{8}{3}} \\
\hline
a^{\frac{8}{3}} & & +a^{\frac{4}{3}}b^{\frac{4}{3}} & & +b^{\frac{8}{3}}
\end{array}$$

9.
$$\begin{array}{llllll}
x^{\frac{5}{4}}-x^{\frac{3}{4}}+x^{\frac{1}{4}}-x^{-\frac{1}{4}} & & & & & \\
x^{\frac{3}{4}}+x^{\frac{1}{4}} & & & & & \\
\hline
x^2 & -x^{\frac{3}{2}} & +x & -x^{\frac{1}{2}} & & \\
 & x^{\frac{3}{2}} & -x & +x^{\frac{1}{2}} & -1 & \\
\hline
x^2 & & & & -1 &
\end{array}$$

10.
$$\begin{array}{lllll}
x^{\frac{7}{4}}-x^{\frac{3}{2}}+x^{\frac{5}{4}}-x & & & & \\
x^{\frac{3}{4}}+x^{\frac{1}{2}} & & & & \\
\hline
x^{\frac{5}{2}} & -x^{\frac{9}{4}} & +x^2 & -x^{\frac{7}{4}} & \\
 & x^{\frac{6}{4}} & -x^2 & +x^{\frac{7}{4}} & -x^{\frac{3}{2}} \\
\hline
x^{\frac{5}{2}} & & & & -x^{\frac{3}{2}}
\end{array}$$

11.

$$\begin{array}{l}
x^n + x^{\frac{n}{2}} + 1 \\
1 + x^{-\frac{n}{2}} + x^{-n} \\
\hline
x^n + x^{\frac{n}{2}} + 1 \\
\quad x^{\frac{n}{2}} + 1 + x^{-\frac{n}{2}} \\
\qquad 1 + x^{-\frac{n}{2}} + x^{-n} \\
\hline
x^n + 2x^{\frac{n}{2}} + 3 + 2x^{-\frac{n}{2}} + x^{-n}
\end{array}$$

12.

$$\begin{array}{l}
x^{2n} + x^n y^n + y^{2n} \\
y^{-2n} + x^{-n}y^{-n} + x^{-2n} \\
\hline
x^{2n}y^{-2n} + x^n y^{-n} + 1 \\
\quad x^n y^{-n} + 1 + x^{-n}y^n \\
\qquad 1 + x^{-n}y^n + x^{-2n}y^{2n} \\
\hline
x^{2n}y^{-2n} + 2x^n y^{-n} + 3 + 2x^{-n}y^n + x^{-2n}y^{2n}
\end{array}$$

13.

$$\begin{array}{l}
a^{\frac{2}{3}} - a^{\frac{1}{3}}b^{\frac{1}{3}} - a^{\frac{1}{3}}c^{\frac{1}{3}} + b^{\frac{2}{3}} - b^{\frac{1}{3}}c^{\frac{1}{3}} + c^{\frac{2}{3}} \\
a^{\frac{1}{3}} + b^{\frac{1}{3}} + c^{\frac{1}{3}} \\
\hline
a - a^{\frac{2}{3}}b^{\frac{1}{3}} - a^{\frac{2}{3}}c^{\frac{1}{3}} + a^{\frac{1}{3}}b^{\frac{2}{3}} - a^{\frac{1}{3}}b^{\frac{1}{3}}c^{\frac{1}{3}} + a^{\frac{1}{3}}c^{\frac{2}{3}} \\
\quad a^{\frac{2}{3}}b^{\frac{1}{3}} - a^{\frac{1}{3}}b^{\frac{2}{3}} - a^{\frac{1}{3}}b^{\frac{1}{3}}c^{\frac{1}{3}} + b - b^{\frac{2}{3}}c^{\frac{1}{3}} + b^{\frac{1}{3}}c^{\frac{2}{3}} \\
\qquad a^{\frac{2}{3}}c^{\frac{1}{3}} - a^{\frac{1}{3}}b^{\frac{1}{3}}c^{\frac{1}{3}} - a^{\frac{1}{3}}c^{\frac{2}{3}} + b^{\frac{2}{3}}c^{\frac{1}{3}} - b^{\frac{1}{3}}c^{\frac{2}{3}} + c \\
\hline
a - 3a^{\frac{1}{3}}b^{\frac{1}{3}}c^{\frac{1}{3}} + b + c
\end{array}$$

14.

$$\begin{array}{l}
\frac{1}{8}a^{\frac{3}{2}} - \frac{1}{12}ab^{\frac{1}{2}} + \frac{1}{18}a^{\frac{1}{2}}b - \frac{1}{27}b^{\frac{3}{2}} \\
\frac{1}{2}a^{\frac{1}{2}} + \frac{1}{3}b^{\frac{1}{2}} \\
\hline
\frac{1}{16}a^2 - \frac{1}{24}a^{\frac{3}{2}}b^{\frac{1}{2}} + \frac{1}{36}ab - \frac{1}{54}a^{\frac{1}{2}}b^{\frac{3}{2}} \\
\quad \frac{1}{24}a^{\frac{3}{2}}b^{\frac{1}{2}} - \frac{1}{36}ab + \frac{1}{54}a^{\frac{1}{2}}b^{\frac{3}{2}} - \frac{1}{81}b^2 \\
\hline
\frac{1}{16}a^2 - \frac{1}{81}b^2
\end{array}$$

15.

$$\begin{array}{l}
81x^{\frac{4}{5}} - 27x^{\frac{3}{5}}y^{\frac{1}{3}} + 9x^{\frac{2}{5}}y^{\frac{2}{3}} - 3x^{\frac{1}{5}}y + y^{\frac{4}{3}} \\
3x^{\frac{1}{5}} + y^{\frac{1}{3}} \\
\hline
243x - 81x^{\frac{4}{5}}y^{\frac{1}{3}} + 27x^{\frac{3}{5}}y^{\frac{2}{3}} - 9x^{\frac{2}{5}}y + 3x^{\frac{1}{5}}y^{\frac{4}{3}} \\
\quad + 81x^{\frac{4}{5}}y^{\frac{1}{3}} - 27x^{\frac{3}{5}}y^{\frac{2}{3}} + 9x^{\frac{2}{5}}y - 3x^{\frac{1}{5}}y^{\frac{4}{3}} + y^{\frac{5}{3}} \\
\hline
243x + y^{\frac{5}{3}}
\end{array}$$

16.
$$\begin{array}{l}
a^{\frac{1}{2}} - b^{\frac{1}{2}}\ )\ a - b\ (\ a^{\frac{1}{2}} + b^{\frac{1}{2}} \\
\underline{a - a^{\frac{1}{2}}b^{\frac{1}{2}}} \\
a^{\frac{1}{2}}b^{\frac{1}{2}} - b \\
a^{\frac{1}{2}}b^{\frac{1}{2}} - b
\end{array}$$

17.
$$\begin{array}{l}
x^{\frac{3}{2}} - y^{\frac{3}{2}}\ )\ x^3 - y^3\ (\ x^{\frac{3}{2}} + y^{\frac{3}{2}} \\
\underline{x^3 - x^{\frac{3}{2}}y^{\frac{3}{2}}} \\
x^{\frac{3}{2}}y^{\frac{3}{2}} - y^3 \\
x^{\frac{3}{2}}y^{\frac{3}{2}} - y^3
\end{array}$$

18.
$$\begin{array}{l}
x^{\frac{n}{2}} - y^{\frac{n}{2}}\ )\ x^{\frac{3n}{2}} - y^{\frac{3n}{2}}\ (\ x^n + x^{\frac{n}{2}}y^{\frac{n}{2}} + y^n \\
\underline{x^{\frac{3n}{2}} - x^n y^{\frac{n}{2}}} \\
x^n y^{\frac{n}{2}} - y^{\frac{3n}{2}} \\
\underline{x^n y^{\frac{n}{2}} - x^{\frac{n}{2}}y^n} \\
x^{\frac{n}{2}}y^n - y^{\frac{3n}{2}} \\
x^{\frac{n}{2}}y^n - y^{\frac{3n}{2}}
\end{array}$$

19.
$$\begin{array}{l}
x^{\frac{2}{3}} + y^{\frac{2}{3}}\ )\ x^2 + y^2\ (\ x^{\frac{4}{3}} - x^{\frac{2}{3}}y^{\frac{2}{3}} + y^{\frac{4}{3}} \\
\underline{x^2 + x^{\frac{4}{3}}y^{\frac{2}{3}}} \\
-x^{\frac{4}{3}}y^{\frac{2}{3}} + y^2 \\
\underline{-x^{\frac{4}{3}}y^{\frac{2}{3}} - x^{\frac{2}{3}}y^{\frac{4}{3}}} \\
x^{\frac{2}{3}}y^{\frac{4}{3}} + y^2 \\
x^{\frac{2}{3}}y^{\frac{4}{3}} + y^2
\end{array}$$

20.
$$\begin{array}{l}
x^{\frac{1}{5}} - 3y^{\frac{1}{3}}\ )\ x - 243y^{\frac{5}{3}}\ (\ x^{\frac{4}{5}} + 3x^{\frac{3}{5}}y^{\frac{1}{3}} + 9x^{\frac{2}{5}}y^{\frac{2}{3}} + 27x^{\frac{1}{5}}y + 81y^{\frac{4}{3}} \\
\underline{x - 3x^{\frac{4}{5}}y^{\frac{1}{3}}} \\
3x^{\frac{4}{5}}y^{\frac{1}{3}} - 243y^{\frac{5}{3}} \\
\underline{3x^{\frac{4}{5}}y^{\frac{1}{3}} - 9x^{\frac{3}{5}}y^{\frac{2}{3}}} \\
9x^{\frac{3}{5}}y^{\frac{2}{3}} - 243y^{\frac{5}{3}} \\
\underline{9x^{\frac{3}{5}}y^{\frac{2}{3}} - 27x^{\frac{2}{5}}y} \\
27x^{\frac{2}{5}}y - 243y^{\frac{5}{3}} \\
\underline{27x^{\frac{2}{5}}y - 81x^{\frac{1}{5}}y^{\frac{4}{3}}} \\
81x^{\frac{1}{5}}y^{\frac{4}{3}} - 243y^{\frac{5}{3}} \\
81x^{\frac{1}{5}}y^{\frac{4}{3}} - 243y^{\frac{5}{3}}
\end{array}$$

21.
$$\begin{array}{l}
y - b^{\frac{1}{2}}y^{\frac{1}{2}} + b\ )\ y^4 + b^2y^2 + b^4\ (\ y^3 + b^{\frac{1}{2}}y^{\frac{5}{2}} - b^{\frac{3}{2}}y^{\frac{3}{2}} + b^{\frac{5}{2}}y^{\frac{1}{2}} + b^3 \\
\underline{y^4 - b^{\frac{1}{2}}y^{\frac{7}{2}} + by^3} \\
b^{\frac{1}{2}}y^{\frac{7}{2}} - by^3 + b^2y^2 + b^4 \\
\underline{b^{\frac{1}{2}}y^{\frac{7}{2}} - by^3 + b^{\frac{3}{2}}y^{\frac{5}{2}}} \\
-b^{\frac{3}{2}}y^{\frac{5}{2}} + b^2y^2 + b^4 \\
\underline{-b^{\frac{3}{2}}y^{\frac{5}{2}} + b^2y^2 - b^{\frac{5}{2}}y^{\frac{3}{2}}} \\
b^{\frac{5}{2}}y^{\frac{3}{2}} + b^4 \\
\underline{b^{\frac{5}{2}}y^{\frac{3}{2}} - b^3y + b^{\frac{7}{2}}y^{\frac{1}{2}}} \\
b^3y - b^{\frac{7}{2}}y^{\frac{1}{2}} + b^4 \\
b^3y - b^{\frac{7}{2}}y^{\frac{1}{2}} + b^4
\end{array}$$

22.

$$\begin{array}{l}
x^{\frac{1}{4}}y^{-\frac{1}{4}}-1+x^{-\frac{1}{4}}y^{\frac{1}{4}}\;)\;x^{\frac{3}{4}}y^{-\frac{3}{4}}+2+x^{-\frac{3}{4}}y^{\frac{3}{4}}\;(\;x^{\frac{1}{2}}y^{-\frac{1}{2}}+x^{\frac{1}{4}}y^{-\frac{1}{4}}+x^{-\frac{1}{4}}y^{\frac{1}{4}}+x^{-\frac{1}{2}}y^{\frac{1}{2}} \\
x^{\frac{3}{4}}y^{-\frac{3}{4}}-x^{\frac{1}{2}}y^{-\frac{1}{2}}+x^{\frac{1}{4}}y^{-\frac{1}{4}} \\ \hline
x^{\frac{1}{2}}y^{-\frac{1}{2}}-x^{\frac{1}{4}}y^{-\frac{1}{4}}+2+x^{-\frac{3}{4}}y^{\frac{3}{4}} \\
x^{\frac{1}{2}}y^{-\frac{1}{2}}-x^{\frac{1}{4}}y^{-\frac{1}{4}}+1 \\ \hline
1+x^{-\frac{3}{4}}y^{\frac{3}{4}} \\
1-x^{-\frac{1}{4}}y^{\frac{1}{4}}+x^{-\frac{1}{2}}y^{\frac{1}{2}} \\ \hline
x^{-\frac{1}{4}}y^{\frac{1}{4}}-x^{-\frac{1}{2}}y^{\frac{1}{2}}+x^{-\frac{3}{4}}y^{\frac{3}{4}} \\
x^{-\frac{1}{4}}y^{\frac{1}{4}}-x^{-\frac{1}{2}}y^{\frac{1}{2}}+x^{-\frac{3}{4}}y^{\frac{3}{4}}
\end{array}$$

23.

$$\begin{array}{l}
a^{\frac{1}{2}}+b^{\frac{1}{3}}\;)\;a^{\frac{5}{2}}+a^{2}b^{\frac{1}{3}}-a^{\frac{3}{2}}b^{\frac{2}{3}}-ab+a^{\frac{1}{2}}b^{\frac{4}{3}}+b^{\frac{5}{3}}\;(\;a^{2}-ab^{\frac{2}{3}}+b^{\frac{4}{3}} \\
a^{\frac{5}{2}}+a^{2}b^{\frac{1}{3}} \\ \hline
-a^{\frac{3}{2}}b^{\frac{2}{3}}-ab+a^{\frac{1}{2}}b^{\frac{4}{3}}+b^{\frac{5}{3}} \\
-a^{\frac{3}{2}}b^{\frac{2}{3}}-ab \\ \hline
a^{\frac{1}{2}}b^{\frac{4}{3}}+b^{\frac{5}{3}} \\
a^{\frac{1}{2}}b^{\frac{4}{3}}+b^{\frac{5}{3}}
\end{array}$$

24.

$$\begin{array}{l}
a^{\frac{2}{3}}-a^{-\frac{2}{3}}\;)\;a^{\frac{4}{3}}-2+a^{-\frac{4}{3}}\;(\;a^{\frac{2}{3}}-a^{-\frac{2}{3}} \\
a^{\frac{4}{3}}-1 \\ \hline
-1+a^{-\frac{4}{3}} \\
-1+a^{-\frac{4}{3}}
\end{array}$$

25. Arrange in descending powers of x. Then

$$\begin{array}{l}
x^{\frac{1}{3}}-x^{-\frac{1}{3}}\;)\;x+2x^{\frac{5}{6}}-2x^{\frac{1}{6}}+2x^{-\frac{1}{6}}-2x^{-\frac{5}{6}}-x^{-1}\;(\;x^{\frac{2}{3}}+2x^{\frac{1}{2}}+1+2x^{-\frac{1}{2}}+x^{-\frac{2}{3}} \\
x-x^{\frac{1}{3}} \\ \hline
2x^{\frac{5}{6}}+x^{\frac{1}{3}}-2x^{\frac{1}{6}}+2x^{-\frac{1}{6}}-2x^{-\frac{5}{6}}-x^{-1} \\
2x^{\frac{5}{6}}-2x^{\frac{1}{6}} \\ \hline
x^{\frac{1}{3}}+2x^{-\frac{1}{6}}-2x^{-\frac{5}{6}}-x^{-1} \\
x^{\frac{1}{3}}-x^{-\frac{1}{3}} \\ \hline
2x^{-\frac{1}{6}}+x^{-\frac{1}{3}}-2x^{-\frac{5}{6}}-x^{-1} \\
2x^{-\frac{1}{6}}-2x^{-\frac{5}{6}} \\ \hline
x^{-\frac{1}{3}}-x^{-1} \\
x^{-\frac{1}{3}}-x^{-1}
\end{array}$$

26.

$$\begin{array}{l}
x^{\frac{1}{3}}y^{-\frac{2}{5}}+x^{-\frac{1}{3}}y^{\frac{2}{5}}\ \big)\ x^{\frac{7}{3}}y^{-\frac{14}{5}}+x^{-\frac{7}{3}}y^{\frac{14}{5}}\ \big(\ x^2y^{-\frac{12}{5}}-x^{\frac{4}{3}}y^{-\frac{8}{5}}+x^{\frac{2}{3}}y^{-\frac{4}{5}}-1+x^{-\frac{2}{3}}y^{\frac{4}{5}}-x^{-\frac{4}{3}}y^{\frac{8}{5}}+x^{-2}y^{\frac{12}{5}} \\
\qquad\qquad\qquad\qquad\underline{x^{\frac{7}{3}}y^{-\frac{14}{5}}+x^{\frac{5}{3}}\,y^{-2}\qquad\quad} \\
\qquad\qquad\qquad\qquad\qquad -x^{\frac{5}{3}}\,y^{-2}+x^{-\frac{7}{3}}y^{\frac{14}{5}} \\
\qquad\qquad\qquad\qquad\qquad \underline{-x^{\frac{5}{3}}\,y^{-2}-x\quad y^{-\frac{6}{5}}\qquad\quad} \\
\qquad\qquad\qquad\qquad\qquad\qquad\quad x\quad y^{-\frac{6}{5}}+x^{-\frac{7}{3}}y^{\frac{14}{5}} \\
\qquad\qquad\qquad\qquad\qquad\qquad\quad \underline{x\quad y^{-\frac{6}{5}}+x^{\frac{1}{3}}\,y^{-\frac{2}{5}}\qquad\quad} \\
\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad -x^{\frac{1}{3}}\,y^{-\frac{2}{5}}+x^{-\frac{7}{3}}y^{\frac{14}{5}} \\
\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \underline{-x^{\frac{1}{3}}\,y^{-\frac{2}{5}}-x^{-\frac{1}{3}}y^{\frac{2}{5}}\qquad\quad} \\
\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad x^{-\frac{1}{3}}y^{\frac{2}{5}}+x^{-\frac{7}{3}}y^{\frac{14}{5}} \\
\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \underline{x^{-\frac{1}{3}}y^{\frac{2}{5}}+x^{-1}\,y^{\frac{6}{5}}\qquad\quad} \\
\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad -x^{-1}\,y^{\frac{6}{5}}+x^{-\frac{7}{3}}y^{\frac{14}{5}} \\
\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \underline{-x^{-1}\,y^{\frac{6}{5}}-x^{-\frac{5}{3}}y^{2}\qquad\quad} \\
\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad x^{-\frac{5}{3}}y^{2}+x^{-\frac{7}{3}}y^{\frac{14}{5}} \\
\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad x^{-\frac{5}{3}}y^{2}+x^{-\frac{7}{3}}y^{\frac{14}{5}}
\end{array}$$

27. $a^{\frac{1}{2}}b^{-\frac{3}{2}}c^{\frac{5}{2}} \times a^{\frac{7}{3}}b^{\frac{4}{3}}c^{-\frac{1}{3}} \times a^{-\frac{5}{6}}b^{\frac{1}{6}}c^{\frac{1}{3}} = a^{\frac{1}{2}+\frac{7}{3}-\frac{5}{6}}b^{-\frac{3}{2}+\frac{4}{3}+\frac{1}{6}}c^{\frac{5}{2}-\frac{1}{3}+\frac{1}{3}} = \&c.$

28.

$$\begin{array}{l}
a + 2a^{\frac{1}{2}}b^{\frac{1}{3}} + 4b^{\frac{2}{3}} \\
\underline{a - 2a^{\frac{1}{2}}b^{\frac{1}{3}} + 4b^{\frac{2}{3}}} \\
a^2 + 2a^{\frac{3}{2}}b^{\frac{1}{3}} + 4ab^{\frac{2}{3}} \\
\qquad - 2a^{\frac{3}{2}}b^{\frac{1}{3}} - 4ab^{\frac{2}{3}} - 8a^{\frac{1}{2}}b \\
\underline{\qquad\qquad\qquad + 4ab^{\frac{2}{3}} + 8a^{\frac{1}{2}}b + 16b^{\frac{4}{3}}} \\
a^2 \qquad\qquad + 4ab^{\frac{2}{3}} \qquad\qquad + 16b^{\frac{4}{3}} \\
\underline{a - 4b^{\frac{2}{3}}} \\
a^3 + 4a^2b^{\frac{2}{3}} + 16ab^{\frac{4}{3}} \\
\underline{\qquad - 4a^2b^{\frac{2}{3}} - 16ab^{\frac{4}{3}} - 64b^2} \\
a^3 \qquad\qquad\qquad - 64b^2
\end{array}$$

29. $\dfrac{(a-a^{-1})(b-b^{-1})}{ab+a^{-1}b^{-1}} = \dfrac{ab-ab^{-1}-a^{-1}b+a^{-1}b^{-1}}{ab+a^{-1}b^{-1}} = 1 - \dfrac{ab^{-1}+a^{-1}b}{ab+a^{-1}b^{-1}}$

$$=1-\frac{(ab^{-1}+a^{-1}b)(ab-a^{-1}b^{-1})}{(ab+a^{-1}b^{-1})(ab-a^{-1}b^{-1})}=1-\frac{a^2+b^2-b^{-2}-a^{-2}}{a^2b^2-a^{-2}b^{-2}}.$$

30. The first expression $=\dfrac{x}{x^{\frac{1}{3}}-1}-\dfrac{1}{x^{\frac{1}{3}}-1}+\dfrac{1}{1+x^{\frac{1}{3}}}-\dfrac{x^{\frac{2}{3}}}{1+x^{\frac{1}{3}}}$

$$=\frac{x-1}{x^{\frac{1}{3}}-1}+\frac{1-x^{\frac{2}{3}}}{1+x^{\frac{1}{3}}}=(x^{\frac{2}{3}}+x^{\frac{1}{3}}+1)+(1-x^{\frac{1}{3}})=\&c.$$

$$\begin{array}{l}
4x^2 - 5x - 4 - 7x^{-1} + 6x^{-2} \\
\underline{3x - 4 + 2x^{-1}} \\
12x^3 - 15x^2 - 12x - 21 + 18x^{-1} \\
\qquad - 16x^2 + 20x + 16 + 28x^{-1} - 24x^{-2} \\
\underline{\qquad\qquad\qquad 8x - 10 - 8x^{-1} - 14x^{-2} + 12x^{-3}}
\end{array}$$

$$\begin{array}{r|l}
3x - 10 + 10x^{-1} - 4x^{-2}\,) & 12x^3 - 31x^2 + 16x - 15 + 38x^{-1} - 38x^{-2} + 12x^{-3}\,(\,4x^2 + 3x + 2 - 3x^{-1} \\
& \underline{12x^3 - 40x^2 + 40x - 16} \\
& \qquad 9x^2 - 24x + 1 + 38x^{-1} \\
& \qquad \underline{9x^2 - 30x + 30 - 12x^{-1}} \\
& \qquad\qquad 6x - 29 + 50x^{-1} - 38x^{-2} \\
& \qquad\qquad \underline{6x - 20 + 20x^{-1} - 8x^{-2}} \\
& \qquad\qquad\quad - 9 + 30x^{-1} - 30x^{-2} + 12x^{-3} \\
& \qquad\qquad\quad - 9 + 30x^{-1} - 30x^{-2} + 12x^{-3}
\end{array}$$

32. $$\frac{1}{1-x^{\frac{1}{4}}}+\frac{1}{1+x^{\frac{1}{4}}}=\frac{1+x^{\frac{1}{4}}+1-x^{\frac{1}{4}}}{(1-x^{\frac{1}{4}})(1+x^{\frac{1}{4}})}=\frac{2}{1-x^{\frac{1}{2}}};$$

then $$\frac{2}{1-x^{\frac{1}{2}}}+\frac{2}{1+x^{\frac{1}{2}}}=\frac{2(1+x^{\frac{1}{2}})+2(1-x^{\frac{1}{2}})}{(1-x^{\frac{1}{2}})(1+x^{\frac{1}{2}})}=\frac{4}{1-x};$$

and $$\frac{4}{1-x}+\frac{4}{1+x}=\frac{8}{1-x^2}.$$

33. $$\begin{array}{l}
(a+b)^{\frac{2}{3}}+(a+b)^{\frac{1}{3}}(a-b)^{\frac{1}{3}}+(a-b)^{\frac{2}{3}}\\
\underline{(a+b)^{\frac{1}{3}}-(a-b)^{\frac{1}{3}}\qquad\qquad\qquad\qquad\qquad\qquad}\\
(a+b)\ +(a+b)^{\frac{2}{3}}(a-b)^{\frac{1}{3}}+(a+b)^{\frac{1}{3}}(a-b)^{\frac{2}{3}}\\
\underline{\qquad\qquad -(a+b)^{\frac{2}{3}}(a-b)^{\frac{1}{3}}-(a+b)^{\frac{1}{3}}(a-b)^{\frac{2}{3}}-(a-b)}\\
(a+b)\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad -(a-b)=2b
\end{array}$$

35. $$\begin{array}{r l}
 & 4x^2a^{-2}-12xa^{-1}+25-24x^{-1}a+16x^{-2}a^2\,(2xa^{-1}-3+4x^{-1}a\\
(2xa^{-1})^2= & 4x^2a^{-2}\\
(2xa^{-1}-3)^2= & \overline{4x^2a^{-2}-12xa^{-1}+9}\\
(2xa^{-1}-3+4x^{-1}a)^2= & \overline{4x^2a^{-2}-12xa^{-1}+25-24x^{-1}a+16x^{-2}a^2}
\end{array}$$

36. $$\begin{array}{r l}
 & 25x^2y^{-2}-20xy^{-1}+9-2x^{-1}y+\frac{1}{4}x^{-2}y^2\left(5xy^{-1}-2+\frac{1}{2}x^{-1}y\right.\\
(5xy^{-1})^2= & 25x^2y^{-2}\\
(5xy^{-1}-2)^2= & \overline{25x^2y^{-2}-20xy^{-1}+4}\\
\left(5xy^{-1}-2+\frac{1}{2}x^{-1}y\right)^2= & \overline{25x^2y^{-2}-20xy^{-1}+9-2x^{-1}y+\frac{1}{4}x^{-2}y^2}
\end{array}$$

37. $$\begin{array}{r l}
 & a^{\frac{8}{5}}+2a^{\frac{4}{5}}x^{\frac{4}{5}}-2a^{\frac{1}{5}}x^{\frac{7}{5}}+x^{\frac{8}{5}}-2a^{-\frac{3}{5}}x^{\frac{11}{5}}+a^{-\frac{6}{5}}x^{\frac{14}{5}}\left(a^{\frac{4}{5}}+x^{\frac{4}{5}}-a^{-\frac{3}{5}}x^{\frac{7}{5}}\right.\\
(a^{\frac{4}{5}})^2= & a^{\frac{8}{5}}\\
(a^{\frac{4}{5}}+x^{\frac{4}{5}})^2= & \overline{a^{\frac{8}{5}}+2a^{\frac{4}{5}}x^{\frac{4}{5}}+\qquad\qquad\quad x^{\frac{8}{5}}}\\
(a^{\frac{4}{5}}+x^{\frac{4}{5}}-a^{-\frac{3}{5}}x^{\frac{7}{5}})^2= & \overline{a^{\frac{8}{5}}+2a^{\frac{4}{5}}x^{\frac{4}{5}}-2a^{\frac{1}{5}}x^{\frac{7}{5}}+x^{\frac{8}{5}}-2a^{-\frac{3}{5}}x^{\frac{11}{5}}+a^{-\frac{6}{5}}x^{\frac{14}{5}}}
\end{array}$$

38. $$\begin{array}{r l}
 & x^{\frac{2}{3}}-4x^{\frac{5}{6}}+4x+2x^{\frac{7}{6}}-4x^{\frac{4}{3}}+x^{\frac{5}{3}}\ \left(\ x^{\frac{1}{3}}-2x^{\frac{1}{2}}+x^{\frac{5}{6}}\right.\\
(x^{\frac{1}{3}})^2= & x^{\frac{2}{3}}\\
(x^{\frac{1}{3}}-2x^{\frac{1}{2}})^2= & \overline{x^{\frac{2}{3}}-4x^{\frac{5}{6}}+4x}\\
(x^{\frac{1}{3}}-2x^{\frac{1}{2}}+x^{\frac{5}{6}})^2= & \overline{x^{\frac{2}{3}}-4x^{\frac{5}{6}}+4x+2x^{\frac{7}{6}}-4x^{\frac{4}{3}}+x^{\frac{5}{3}}}
\end{array}$$

39. (i) $(x^{\frac{3}{4}}-1)^2=0$; $\therefore\ x^{\frac{3}{4}}=1$; $\therefore\ x^3=1$;

$$\therefore\ x=1 \text{ or } -\frac{1}{2}\pm\sqrt{-\frac{3}{4}}.\quad \text{[Ex. 2, Art. 155.]}$$

(ii) $(x^{\frac{1}{2}}-2)(x^{\frac{1}{2}}-6)=0$; $\therefore\ x^{\frac{1}{2}}=2$, or $x^{\frac{1}{2}}=6$. Hence

$x=4$ or $x=36$.

(iii) $(x^{\frac{3}{4}}-27)(x^{\frac{3}{4}}+1)=0$. If $x^{\frac{3}{4}}=27$, $x=27^{\frac{4}{3}}=27\times\sqrt[3]{27}$, and

$$\sqrt[3]{27}=3 \text{ or } 3\left\{-\frac{1}{2}\pm\sqrt{-\frac{3}{4}}\right\}.$$

If $x^{\frac{3}{4}}=-1$, $x=\sqrt[3]{1}=1$ or $-\frac{1}{2}\pm\sqrt{-\frac{3}{4}}$.

(iv) $x^{\frac{1}{3}}-4+3x^{-\frac{1}{3}}=0$; $\therefore$ $x^{\frac{2}{3}}-4x^{\frac{1}{3}}+3=0$, *i. e.* $(x^{\frac{1}{3}}-3)(x^{\frac{1}{3}}-1)=0$. Hence $x^{\frac{1}{3}}=3$ and $x=27$, or $x^{\frac{1}{3}}=1$ and $x=1$.

(v) $4x^{\frac{1}{5}}-4-3x^{-\frac{1}{5}}=0$; $\therefore$ $4x^{\frac{2}{5}}-4x^{\frac{1}{5}}-3=0$; $\therefore$ $x^{\frac{1}{5}}=\frac{3}{2}$ or $-\frac{1}{2}$;

$$\therefore x=\frac{243}{32} \text{ or } -\frac{1}{32}.$$

LVIII. Pages 234—235.

1. $\sqrt{9}\times\sqrt{3}+\sqrt{16}\times\sqrt{3}=3\sqrt{3}+4\sqrt{3}=7\sqrt{3}$

2. $\sqrt{25}\times\sqrt{2}+\sqrt{49}\times\sqrt{2}=5\sqrt{2}+7\sqrt{2}=12\sqrt{2}$.

3. $\sqrt{9}\times\sqrt{5}+2\sqrt{25}\times\sqrt{5}=3\sqrt{5}+10\sqrt{5}=13\sqrt{5}$.

4. $2\sqrt{36}\times\sqrt{5}-\sqrt{81}\times\sqrt{5}=12\sqrt{5}-9\sqrt{5}=3\sqrt{5}$.

5. $2\sqrt{4}\times\sqrt{7}-\sqrt{9}\times\sqrt{7}=4\sqrt{7}-3\sqrt{7}=\sqrt{7}$.

6. $5\sqrt{16}\times\sqrt{13}-3\sqrt{25}\times\sqrt{13}=20\sqrt{13}-15\sqrt{13}=5\sqrt{13}$.

7. $3\sqrt[3]{5}+\sqrt[3]{125}\times\sqrt[3]{5}=3\sqrt[3]{5}+5\sqrt[3]{5}=8\sqrt[3]{5}$.

8. $3\sqrt[3]{8}\times\sqrt[3]{9}-2\sqrt[3]{27}\times\sqrt[3]{9}=6\sqrt[3]{9}-6\sqrt[3]{9}=0$.

9. $4\sqrt[3]{64}\times\sqrt[3]{7}-15\sqrt[3]{7}=16\sqrt[3]{7}-15\sqrt[3]{7}=\sqrt[3]{7}$.

10. $\sqrt{256}\times\sqrt{2}-\sqrt{25}\times\sqrt{2}-\sqrt{49}\sqrt{2}=16\sqrt{2}-5\sqrt{2}-7\sqrt{2}=4\sqrt{2}$.

11. $3\sqrt{4}\times\sqrt{3}-\sqrt{9}\times\sqrt{3}+2\sqrt{25}\times\sqrt{3}=6\sqrt{3}-3\sqrt{3}+10\sqrt{3}=13\sqrt{3}$.

12. $\sqrt{49}\times\sqrt{3}-2\sqrt{9}\times\sqrt{3}-\sqrt{3}=7\sqrt{3}-6\sqrt{3}-\sqrt{3}=0$.

13. $$5\times\frac{1}{\sqrt{2}}-\sqrt{4}\times\sqrt{2}+\frac{1}{\sqrt{9}\times\sqrt{2}}=\frac{5}{\sqrt{2}}-2\sqrt{2}+\frac{1}{3\sqrt{2}}$$
$$=\frac{5\sqrt{2}}{2}-2\sqrt{2}+\frac{\sqrt{2}}{3\times 2}=\frac{2}{3}\sqrt{2}.$$

14. $\sqrt{15\times 60}=\sqrt{900}=30$.

15. $\sqrt{12\times 24}=\sqrt{12\times 12\times 2}=\sqrt{12^2}\times\sqrt{2}=12\sqrt{2}$.

16. $2\times 3\sqrt{\frac{2}{3}\times\frac{3}{2}}=6$.

17. $\sqrt[3]{6\times 36}=\sqrt[3]{6^3}=6$.

18. $\sqrt{12\times 27\times 75}=\sqrt{12\times 3\times 9\times 25\times 3}=\sqrt{36\times 9\times 25\times 3}$
$=\sqrt{36}\times\sqrt{9}\times\sqrt{25}\times\sqrt{3}=6\times 3\times 5\sqrt{3}=90\sqrt{3}.$

19. $\sqrt[3]{16\times 6\times 9}=\sqrt[3]{8\times 2\times 2\times 3\times 9}=\sqrt[3]{8\times 27\times 4}=\sqrt[3]{8}\times\sqrt[3]{27}\times\sqrt[3]{4}=6\sqrt[3]{4}.$

20. $\sqrt[3]{12\times 75\times 30}=\sqrt[3]{3\times 2^2\times 3\times 5^2\times 2\times 3\times 5}=\sqrt[3]{3^3 . 2^3 . 5^3}=3\times 2\times 5=30.$

21. $\sqrt[4]{6\times 12\times 18}=\sqrt[4]{6\times 6\times 2\times 3\times 6}=\sqrt[4]{6\times 6\times 6\times 6}=6.$

22. $\sqrt[6]{10^3}\times\sqrt[6]{200^2}=\sqrt[6]{(10^3\times 200^2)}=\sqrt[6]{(10^6\times 40)}=10\sqrt[6]{40}.$

23. $\sqrt[6]{4^2}\times\sqrt[6]{8^3}=\sqrt[6]{(4^2\times 8^3)}=\sqrt[6]{(4^6\times 2)}=4\sqrt[6]{2}.$

24. $\sqrt{\frac{18}{50}}=\sqrt{\frac{9}{25}}=\frac{3}{5}.$ 25. $\sqrt{\frac{63}{112}}=\sqrt{\frac{9}{16}}=\frac{3}{4}.$

26. $\sqrt{\frac{20\times 96}{30}}=\sqrt{64}=8.$

27. $\sqrt[3]{\frac{147\times 735}{35}}=\sqrt[3]{(147\times 21)}=\sqrt[3]{(7^3\times 9)}=7\sqrt[3]{9}.$

28. $\sqrt{3}=\sqrt[6]{3^3}=\sqrt[6]{27}$, and $\sqrt[3]{(5\tfrac{1}{5})}=\sqrt[6]{(5\tfrac{1}{5})^2}=\sqrt[6]{(27\tfrac{1}{25})}.$

29. $\sqrt{50}=\sqrt[12]{50^6}=\sqrt[12]{15625000000}.$ $\sqrt[3]{344}=\sqrt[12]{344^4}=\sqrt[12]{14003408896}.$
$\sqrt[4]{2402}=\sqrt[12]{2402^3}=\sqrt[12]{13858588808}$

30.
$$\begin{array}{lll}
\sqrt{6}- & \sqrt{3} & \\
\sqrt{6}+ & \sqrt{3} & \\ \hline
6-3\sqrt{2} & & \\
 & 3\sqrt{2}-3 & \\ \hline
6 & & -3=3
\end{array}$$

31.
$$\begin{array}{lll}
2\sqrt{5}+3\sqrt{3} & & \\
3\sqrt{5}-4\sqrt{3} & & \\ \hline
6\times 5+9\sqrt{15} & & \\
 & -8\sqrt{15} & -12\times 3 \\ \hline
30+ & \sqrt{15} & -36=\sqrt{15}-6
\end{array}$$

32.
$$\begin{array}{lllll}
\sqrt{2}+ & \sqrt{3}+\sqrt{6} & & & \\
2\sqrt{2}+ & 3\sqrt{3}+\sqrt{6} & & & \\ \hline
2\times 2+ & 2\sqrt{6}+2\times 2\sqrt{3} & & & \\
 & 3\sqrt{6} & & +3\times 3+3\times 3\sqrt{2} & \\
 & & 2\sqrt{3} & + & 3\sqrt{2}+6 \\ \hline
4+ & 5\sqrt{6}+ & 6\sqrt{3}+9 & + & 12\sqrt{2}+6
\end{array}$$

33.
$$\begin{array}{llll}
1+ & \sqrt[3]{3}+\sqrt[3]{5} & & \\
1+ & \sqrt[3]{3}-\sqrt[3]{5} & & \\ \hline
1+ & \sqrt[3]{3}+\sqrt[3]{5} & & \\
+ & \sqrt[3]{3} & +\sqrt[3]{9}+\sqrt[3]{15} & \\
 & -\sqrt[3]{5} & -\sqrt[3]{15} & -\sqrt[3]{25} \\ \hline
1+2\sqrt[3]{3} & & +\sqrt[3]{9} & -\sqrt[3]{25}
\end{array}$$

34. $\{(\sqrt{2}+\sqrt{3})+\sqrt{5}\}^2=2+3+5+2\sqrt{2}\sqrt{3}+2\sqrt{2}\sqrt{5}+2\sqrt{3}\sqrt{5}=$&c.

35. The continued product is
$\{(\sqrt{3}+\sqrt{5})+\sqrt{2}\}\{(\sqrt{3}+\sqrt{5})-\sqrt{2})\}\{\sqrt{2}-(\sqrt{3}-\sqrt{5})\}\{\sqrt{2}+(\sqrt{3}-\sqrt{5})\}$
$=\{(\sqrt{3}+\sqrt{5})^2-2\}\{2-(\sqrt{3}-\sqrt{5})^2\}=\{2\sqrt{15}+6\}\{2\sqrt{15}-6\}=60-36.$

LIX. PAGES 238—240.

1. $\frac{3\times\sqrt{7}}{\sqrt{7}\times\sqrt{7}}=\frac{3}{7}\sqrt{7}.$

2. $\frac{2\times\sqrt{5}}{\sqrt{5}\times\sqrt{5}}=\frac{2}{5}\sqrt{5}.$

3. $\frac{3\sqrt{2}\times\sqrt{3}}{2\sqrt{3}\times\sqrt{3}}=\frac{3\sqrt{6}}{6}=\frac{1}{2}\sqrt{6}.$

4. $\frac{\sqrt{6}\times\sqrt{5}}{\sqrt{5}\times\sqrt{5}}=\frac{1}{5}\sqrt{30}.$

5. $\frac{2(\sqrt{2}-1)}{(\sqrt{2}+1)(\sqrt{2}-1)}=\frac{2\sqrt{2}-2}{2-1}=2\sqrt{2}-2.$

6. $\frac{(\sqrt{2}-1)^2}{(\sqrt{2}+1)(\sqrt{2}-1)}=\frac{2-2\sqrt{2}+1}{2-1}=3-2\sqrt{2}.$

7. $\frac{2\sqrt{5}(\sqrt{5}-\sqrt{3})}{(\sqrt{5}+\sqrt{3})(\sqrt{5}-\sqrt{3})}=\frac{10-2\sqrt{15}}{5-3}=5-\sqrt{15}.$

8. $$\frac{(15+14\sqrt{3})(15+2\sqrt{3})}{(15-2\sqrt{3})(15+2\sqrt{3})}=\frac{225+30\sqrt{3}+210\sqrt{3}+84}{225-12}$$
$$=\frac{309+240\sqrt{3}}{213}=\frac{103+80\sqrt{3}}{71}.$$

9. $$\frac{(\sqrt{5}+3\sqrt{3})(2\sqrt{5}+\sqrt{3})}{(2\sqrt{5}-\sqrt{3})(2\sqrt{5}+\sqrt{3})}=\frac{10+\sqrt{15}+6\sqrt{15}+9}{20-3}=\frac{19+7\sqrt{15}}{17}.$$

10. $$\frac{(\sqrt{6}-3\sqrt{12})(2\sqrt{6}-\sqrt{12})}{(2\sqrt{6}+\sqrt{12})(2\sqrt{6}-\sqrt{12})}=\frac{12-6\sqrt{2}-36\sqrt{2}+36}{24-12}$$
$$=\frac{48-42\sqrt{2}}{12}=4-\frac{7}{2}\sqrt{2}.$$

11. $$\frac{(1+\sqrt{2}-\sqrt{3})}{(1+\sqrt{2}+\sqrt{3})(1+\sqrt{2}-\sqrt{3})}=\frac{1+\sqrt{2}-\sqrt{3}}{(1+\sqrt{2})^2-3}=\frac{1+\sqrt{2}-\sqrt{3}}{1+2\sqrt{2}+2-3}$$
$$=\frac{1+\sqrt{2}-\sqrt{3}}{2\sqrt{2}}=\frac{(1+\sqrt{2}-\sqrt{3})\sqrt{2}}{2\sqrt{2}\times\sqrt{2}}=\frac{1}{4}(\sqrt{2}+2-\sqrt{6}).$$

12. $$\frac{3(2+\sqrt{3}-\sqrt{5})}{(2+\sqrt{3}+\sqrt{5})(2+\sqrt{3}-\sqrt{5})}=\frac{6+3\sqrt{3}-3\sqrt{5}}{(2+\sqrt{3})^2-5}=\frac{6+3\sqrt{3}-3\sqrt{5}}{4+4\sqrt{3}+3-5}$$
$$=\frac{6+3\sqrt{3}-3\sqrt{5}}{2(2\sqrt{3}+1)}=\frac{(6+3\sqrt{3}-3\sqrt{5})(2\sqrt{3}-1)}{2(2\sqrt{3}+1)(2\sqrt{3}-1)}$$
$$=\frac{12\sqrt{3}-6+18-3\sqrt{3}-6\sqrt{15}+3\sqrt{5}}{2(12-1)}=\&c.$$

13. $$\frac{(\sqrt[3]{2})^2+\sqrt[3]{2}+1}{(\sqrt[3]{2}-1)\{(\sqrt[3]{2})^2+\sqrt[3]{2}+1\}}+\frac{3\{(\sqrt[3]{2})^2-\sqrt[3]{2}+1\}}{(\sqrt[3]{2}+1)\{(\sqrt[3]{2})^2-\sqrt[3]{2}+1\}}$$
$$=\frac{\sqrt[3]{4}+\sqrt[3]{2}+1}{(\sqrt[3]{2})^3-1}+\frac{3\{\sqrt[3]{4}-\sqrt[3]{2}+1\}}{(\sqrt[3]{2})^3+1}=(\sqrt[3]{4}+\sqrt[3]{2}+1)+(\sqrt[3]{4}-\sqrt[3]{2}+1)=2+2\sqrt[3]{4}.$$

14. $$\frac{4\{(\sqrt[3]{9})^2+\sqrt[3]{9}+1\}}{(\sqrt[3]{9}-1)\{(\sqrt[3]{9})^2+(\sqrt[3]{9})+1\}}+\frac{5\{(\sqrt[3]{9})^2-\sqrt[3]{9}+1\}}{(\sqrt[3]{9}+1)\{(\sqrt[3]{9})^2-\sqrt[3]{9}+1\}}$$

$$=\frac{4\{\sqrt[3]{81}+\sqrt[3]{9}+1\}}{(\sqrt[3]{9})^3-1}+\frac{5\{\sqrt[3]{81}-\sqrt[3]{9}+1\}}{(\sqrt[3]{9})^3+1}$$

$$=\frac{4\{\sqrt[3]{81}+\sqrt[3]{9}+1\}}{9-1}+\frac{5\{\sqrt[3]{81}-\sqrt[3]{9}+1\}}{9+1}=\&\text{c.}$$

15. $$\frac{(2+\sqrt{3})^2+(2-\sqrt{3})^2}{(2-\sqrt{3})^2\,(2+\sqrt{3})^2}=\frac{(4+4\sqrt{3}+3)+(4-4\sqrt{3}+3)}{\{(2-\sqrt{3})\,(2+\sqrt{3})\}^2}=14.$$

16. $$\frac{(2+\sqrt{3})^3+(2-\sqrt{3})^3}{(2-\sqrt{3})^3\,(2+\sqrt{3})^3}=\frac{(8+12\sqrt{3}+18+3\sqrt{3})+(8-12\sqrt{3}+18-3\sqrt{3})}{\{(2-\sqrt{3})\,(2+\sqrt{3})\}^3}$$

$$=\frac{52}{\{4-3\}^3}=52.$$

17. $$\frac{(3+\sqrt{2})^2\,(5-\sqrt{2})^2}{[(3-\sqrt{2})\,(3+\sqrt{2})][(5+\sqrt{2})\,(5-\sqrt{2})]}=\frac{\{(3+\sqrt{2})\,(5-\sqrt{2})\}^2}{[9-2]\,[25-2]}$$

$$=\frac{\{13+2\sqrt{2}\}^2}{7\times 23}=\frac{169+52\sqrt{2}+8}{161}.$$

18. $$\frac{(2\sqrt{15}-3\sqrt{5}+2\sqrt{2})\,(\sqrt{15}-\sqrt{2})}{(\sqrt{15}+\sqrt{2})\,(\sqrt{15}-\sqrt{2})}=\frac{30-2\sqrt{30}-15\sqrt{3}+3\sqrt{10}+2\sqrt{30}-4}{15-2}$$

$$=\&\text{c.}$$

19. $$\frac{(5+\sqrt{7})\,\{(7-2\sqrt{5})\,(31+13\sqrt{5})\}}{\{(6-2\sqrt{7})\,(11+4\sqrt{7})\}(3+\sqrt{5})}=\frac{(5+\sqrt{7})\,\{217+91\sqrt{5}-62\sqrt{5}-130\}}{\{66+24\sqrt{7}-22\sqrt{7}-56\}\,(3+\sqrt{5})}$$

$$=\frac{(5+\sqrt{7})\,\{87+29\sqrt{5}\}}{\{10+2\sqrt{7}\}\,(3+\sqrt{5})}=\frac{29\,(5+\sqrt{7})\,(3+\sqrt{5})}{2\,(5+\sqrt{7})\,(3+\sqrt{5})}=\frac{29}{2}.$$

20. $$\frac{\sqrt{3}+\sqrt{2}}{\sqrt{3}-\sqrt{2}}=\frac{(\sqrt{3}+\sqrt{2})^2}{(\sqrt{3}-\sqrt{2})\,(\sqrt{3}+\sqrt{2})}=\frac{3+2\sqrt{6}+2}{3-2}=5+2\sqrt{6}.$$

$$\frac{\sqrt{3}-\sqrt{2}}{\sqrt{3}+\sqrt{2}}=\frac{(\sqrt{3}-\sqrt{2})^2}{(\sqrt{3}+\sqrt{2})\,(\sqrt{3}-\sqrt{2})}=\frac{3-2\sqrt{6}+2}{3-2}=5-2\sqrt{6}.$$

Thus we have

$$(5+2\sqrt{6})^2+(5-2\sqrt{6})^2=25+20\sqrt{6}+24+25-20\sqrt{6}+24=98.$$

21. $$\frac{1+\sqrt{3}+\sqrt{6}}{1+\sqrt{2}-\sqrt{3}}=\frac{(1+\sqrt{3}+\sqrt{6})\,(1+\sqrt{2}+\sqrt{3})}{(1+\sqrt{2}-\sqrt{3})\,(1+\sqrt{2}+\sqrt{3})}$$

$$=\frac{1}{2\sqrt{2}}\,\{4+4\sqrt{2}+4\sqrt{3}+2\sqrt{6}\}=\sqrt{2}+2+\sqrt{6}+\sqrt{3}.$$

22. $$\frac{(-1+\sqrt{2}+\sqrt{3})+(1+\sqrt{2}+\sqrt{3})}{(1+\sqrt{2}+\sqrt{3})\,(-1+\sqrt{2}+\sqrt{3})}+\frac{(1+\sqrt{2}-\sqrt{3})+(1-\sqrt{2}+\sqrt{3})}{\{1-(\sqrt{2}-\sqrt{3})\}\{1+(\sqrt{2}-\sqrt{3})\}}$$

$$=\frac{2\sqrt{2}+2\sqrt{3}}{4+2\sqrt{6}}+\frac{2}{2\sqrt{6}-4}=\frac{\sqrt{2}+\sqrt{3}}{\sqrt{6}+2}+\frac{1}{\sqrt{6}-2}$$

$$=\frac{(\sqrt{6}-2)\,(\sqrt{2}+\sqrt{3})+\sqrt{6}+2}{(\sqrt{6}+2)\,(\sqrt{6}-2)}=\frac{1}{2}(\sqrt{2}+\sqrt{6}+2).$$

23. $$\frac{3\sqrt{2}\,(\sqrt{6}-\sqrt{3})}{(\sqrt{6}+\sqrt{3})\,(\sqrt{6}-\sqrt{3})}-\frac{4\sqrt{3}\,(\sqrt{6}-\sqrt{2})}{(\sqrt{6}+\sqrt{2})\,(\sqrt{6}-\sqrt{2})}+\frac{\sqrt{6}\,(\sqrt{3}-\sqrt{2})}{(\sqrt{3}+\sqrt{2})\,(\sqrt{3}-\sqrt{2})}$$

$$=\frac{6\sqrt{3}-3\sqrt{6}}{6-3}-\frac{12\sqrt{2}-4\sqrt{6}}{6-2}+\frac{3\sqrt{2}-2\sqrt{3}}{3-2}$$

$$=(2\sqrt{3}-\sqrt{6})-(3\sqrt{2}-\sqrt{6})+(3\sqrt{2}-2\sqrt{3})=0.$$

24. $$\left(\frac{2-\sqrt{3}}{2}\right)^{-\frac{3}{2}}=\left(\frac{2}{2-\sqrt{3}}\right)^{\frac{3}{2}}=(4+2\sqrt{3})^{\frac{3}{2}};$$

and $$\left(\frac{2+\sqrt{3}}{2}\right)^{\frac{3}{2}}=\left\{\frac{4+2\sqrt{3}}{4}\right\}^{\frac{3}{2}}=\frac{1}{8}(4+2\sqrt{3})^{\frac{3}{2}}.$$

Hence we have $$\frac{7}{8}(4+2\sqrt{3})^{\frac{3}{2}}=\frac{7}{8}\{(\sqrt{3}+1)^2\}^{\frac{3}{2}}$$

$$=\frac{7}{8}(\sqrt{3}+1)^3=7\left(\frac{\sqrt{3}+1}{2}\right)^3.$$

25. $$\frac{\{2\sqrt{(a+b)}+3\sqrt{(a-b)}\}\,\{2\sqrt{(a+b)}+\sqrt{(a-b)}\}}{\{2\sqrt{(a+b)}-\ \sqrt{(a-b)}\}\,\{2\sqrt{(a+b)}+\sqrt{(a-b)}\}}$$

$$=\frac{4\,(a+b)+8\sqrt{(a^2-b^2)}+3\,(a-b)}{4\,(a+b)-(a-b)}=\&\text{c}.$$

26. $$\frac{(\sqrt{2}+1)^2}{(\sqrt{2}-1)\,(\sqrt{2}+1)}=\frac{3+2\sqrt{2}}{2-1}=3+2\sqrt{2}=3+2\times 1{\cdot}4142=5{\cdot}828\ldots$$

27. $$\frac{\sqrt{3}\,(2+\sqrt{3})}{(2-\sqrt{3})\,(2+\sqrt{3})}+\frac{(2-\sqrt{2})^2}{(2+\sqrt{2})\,(2-\sqrt{2})}=\frac{2\sqrt{3}+3}{4-3}+\frac{6-4\sqrt{2}}{4-2}$$

$$=(2\sqrt{3}+3)+(3-2\sqrt{2})=6+2\sqrt{3}-2\sqrt{2}$$

$$=6+2\times 1{\cdot}7320\ldots-2\times 1{\cdot}4142\ldots=6{\cdot}636.$$

28. Let $\sqrt{(6+\sqrt{20})}=\sqrt{x}+\sqrt{y}$. Then, as in Art. 191, $x+y=6$ and $2\sqrt{xy}=\sqrt{20}$. Thus $x+y=6$ and $xy=5$, &c.

29. Let $\sqrt{16+6\sqrt{7}}=\sqrt{x}+\sqrt{y}$. Then $x+y=16$ and $2\sqrt{xy}=6\sqrt{7}$. Thus $x+y=16$ and $xy=63$, &c.

30. Let $\sqrt{12-6\sqrt{3}}=\sqrt{x}-\sqrt{y}$. Then $x+y=12$ and $2\sqrt{xy}=6\sqrt{3}$. Thus $x+y=12$ and $xy=27$, &c.

31. Let $\sqrt{28-5\sqrt{12}}=\sqrt{x}-\sqrt{y}$. Then $x+y=28$, $2\sqrt{xy}=5\sqrt{12}$. Thus $x+y=28$, $xy=75$, &c.

32. Let $\sqrt{101-28\sqrt{13}}=\sqrt{x}-\sqrt{y}$. Then $x+y=101$, $2\sqrt{xy}=28\sqrt{13}$. Thus $x+y=101$, $xy=2548$, &c.

33. $\sqrt{117+36\sqrt{10}}=3\sqrt{13+4\sqrt{10}}=3(\sqrt{x}+\sqrt{y})$. Then $x+y=13$ and
$2\sqrt{xy}=4\sqrt{10}$; $\therefore\ xy=40$, &c.

34. $\sqrt{280+56\sqrt{21}}=\sqrt{28}\sqrt{10+2\sqrt{21}}$. Let $\sqrt{10+2\sqrt{21}}=\sqrt{x}+\sqrt{y}$. Then $x+y=10$ and $xy=21$. Hence $x=7$ and $y=3$. Thus

$$\sqrt{280+56\sqrt{21}}=2\sqrt{7}\,(\sqrt{7}+\sqrt{3})=14+2\sqrt{21}.$$

35. $\sqrt{\left(\frac{9}{2}+2\sqrt{2}\right)}=\sqrt{x}+\sqrt{y}$. Then $x+y=\frac{9}{2}$, $xy=2$; $\therefore\ x=4,\ y=\frac{1}{2}$.

36. $3\sqrt{5}-\sqrt{2}+(\sqrt{5}+\sqrt{2})=4\sqrt{5}$.

37. $6-4\sqrt{3}+2\sqrt{(4-2\sqrt{3})}=6-4\sqrt{3}+2\,(\sqrt{3}-1)=4-2\sqrt{3}$.

38. $$\begin{aligned}11+2\,(1+\sqrt{5})\,(1+\sqrt{7})&=13+2\sqrt{5}+2\sqrt{7}+2\sqrt{35}\\&=1+5+7+2\sqrt{5}+2\sqrt{7}+2\sqrt{5}\sqrt{7}=(1+\sqrt{5}+\sqrt{7})^2.\end{aligned}$$

39. $$\begin{aligned}2x+2\sqrt{x^2-1}&=(x+1)+(x-1)+2\sqrt{x+1}\sqrt{x-1}\\&=(\sqrt{x+1}+\sqrt{x-1})^2.\end{aligned}$$

40. $$\begin{aligned}2a+2\sqrt{a^2-x^2}&=(a+x)+(a-x)+2\sqrt{(a+x)}\sqrt{(a-x)}\\&=\{\sqrt{(a+x)}+\sqrt{(a-x)}\}^2.\end{aligned}$$

41. $$\begin{aligned}3x-1+2\sqrt{(2x^2+x-6)}&=(2x-3)+(x+2)+2\sqrt{(2x-3)}\sqrt{(x+2)}\\&=\{\sqrt{(2x-3)}+\sqrt{(x+2)}\}^2.\end{aligned}$$

42. $\dfrac{1}{\sqrt{(16+6\sqrt{7})}}=\dfrac{1}{3+\sqrt{7}}=\dfrac{3-\sqrt{7}}{2}$.

43. $\dfrac{1}{\sqrt{(15+2\sqrt{56})}}=\dfrac{1}{\sqrt{8}+\sqrt{7}}=\sqrt{8}-\sqrt{7}$.

44. $(\sqrt{5}+\sqrt{2})+(\sqrt{5}-\sqrt{2})=2\sqrt{5}$.

45. $\dfrac{(1+\sqrt{2})-\sqrt{2}}{\sqrt{2}+(\sqrt{2}-1)}=\dfrac{1}{2\sqrt{2}-1}=\dfrac{1}{7}(2\sqrt{2}+1)$.

46. First $\sqrt{(2+\sqrt{3})}=\dfrac{\sqrt{(4+2\sqrt{3})}}{\sqrt{2}}=\dfrac{\sqrt{3}+1}{\sqrt{2}}$, and $\sqrt{(2-\sqrt{3})}=\dfrac{\sqrt{3}-1}{\sqrt{2}}$.

Hence we have $\dfrac{2+\sqrt{3}}{\sqrt{2}+\dfrac{\sqrt{3}+1}{\sqrt{2}}}-\dfrac{2-\sqrt{3}}{\sqrt{2}-\dfrac{\sqrt{3}-1}{\sqrt{2}}}=\dfrac{2\sqrt{2}+\sqrt{6}}{3+\sqrt{3}}-\dfrac{2\sqrt{2}-\sqrt{6}}{3-\sqrt{3}}=\dfrac{1}{3}\sqrt{6}$.

47. $\dfrac{\sqrt{2}+\sqrt{45}}{\sqrt{2}+(\sqrt{5}-\sqrt{2})}=\dfrac{\sqrt{2}+\sqrt{45}}{\sqrt{5}}=3+\dfrac{1}{5}\sqrt{10}=$&c.

48. If $x=2+\sqrt{5}$, $x-2=\sqrt{5}$; $\therefore\ x^2-4x+4=5$; $\therefore\ x^2-4x+5=6$.

Or, $x^2-4x+5=(2+\sqrt{5})^2-4(2+\sqrt{5})+5=9+4\sqrt{5}-(8+4\sqrt{5})+5=6$.

49. If $x^2=x+1$; then $x=\frac{1}{2}\pm\frac{1}{2}\sqrt{5}$. Hence

$$x^3=\frac{1}{8}(1\pm\sqrt{5})^3=\frac{1}{8}(1\pm 3\sqrt{5}+15\pm 5\sqrt{5})$$
$$=2\pm\sqrt{5}=2\left(\frac{1}{2}\pm\frac{1}{2}\sqrt{5}\right)+1=2x+1.$$

The second part can be proved in a similar manner.

Or thus. We have $x^2=x+1$; $\therefore\ x^3=x^2+x=x+1+x=2x+1$;

$$\therefore\ x^4=2x^2+x=2(x+1)+x=3x+2;$$
$$\therefore\ x^5=3x^2+2x=3(x+1)+2x=5x+3.$$

50. We have $x^2=3x+5$;

$$\therefore\ x^3=3x^2+5x=3(3x+5)+5x=14x+15;$$
$$\therefore\ x^4=14x^2+15x=14(3x+5)+15x=57x+70.$$

LX. PAGE 245.

1. 5 : 6, 7 : 8, 41 : 48 and 31 : 36 are equivalent respectively to 120 : 144, 126 : 144, 123 : 144 and 124 : 144.

2. If $3+x : 4+x=5 : 6$, $6(3+x)=5(4+x)$, whence $x=2$.

3. If $15+x : 17+x=1 : 2$, $2(15+x)=17+x$, whence $x=-13$.

4. $3+x : 4+x=25 : 32$. Hence $32(3+x)=25(4+x)$.

5. $x : y=5 : 6$; $\therefore\ 6x=5y$. Also $x+y=121$, &c.

6. $x : y=3 : 8$; $\therefore\ 8x=3y$. Also $x^2+y^2=3577$, &c.

7. $4x+5y=6x-2y$; $\therefore\ 2x=7y$; $\therefore\ x : y=7 : 2$.

8. $4x^2-4xy+y^2=0$; $\therefore\ 2x-y=0$; $\therefore\ x : y=1 : 2$.

9. $x^2-5xy+6y^2=0$, *i.e.* $(x-3y)(x-2y)=0$;

$$\therefore\ x : y=3 : 1 \text{ or } x : y=2 : 1.$$

10. Let $x : y$ be the ratio. Then

$$x+2 : y+2=2 : 3, \text{ and } x-1 : y-1=1 : 2.$$

Hence $3(x+2)=2(y+2)$ and $2(x-1)=y-1$, &c.

11. If x and y be the numbers we have $\frac{x+y}{7}=\frac{x-y}{1}=\frac{x^2+y^2}{75}$.

From $x+y=7(x-y)$; $3x=4y$. Hence $75(x+y)=7(x^2+y^2)$ becomes

$$75\left(\frac{4}{3}y+y\right)=7\left(\frac{16}{9}y^2+y^2\right);$$
$$\therefore\ y=0 \text{ or } y=9. \text{ Then } x=12.$$

12. $\frac{9+x}{23+x} > \frac{7}{11}$, if $11(9+x) > 7(23+x)$;

that is, if $4x > 62$ or $x > 15\frac{1}{2}$.

15. $x+1 : x+4 = 3^2 : 5^2$, &c.

16. Let $3x$ and $4x$ be their present ages. Their ages 30 years ago were $3x-30$ and $4x-30$. Hence

$$3x-30 : 4x-30 = 1 : 3; \quad \therefore 3(3x-30) = 4x-30, \text{ \&c.}$$

17. Let $x : y$ be the original ratio. Then

$$x - \frac{1}{y} : y - \frac{1}{x} = \frac{xy-1}{y} : \frac{xy-1}{x} = \frac{1}{y} : \frac{1}{x} = x : y.$$

18. $$\frac{a^2-x^2}{a^2+x^2} - \frac{a-x}{a+x} = \frac{(a^2-x^2)(a+x) - (a^2+x^2)(a-x)}{(a^2+x^2)(a+x)}$$

$$= \frac{2a^2x - 2ax^2}{(a^2+x^2)(a+x)} = \frac{2ax(a-x)}{(a^2+x^2)(a+x)};$$

thus, if a and x be positive, and a greater than x, then $\frac{a^2-x^2}{a^2+x^2} - \frac{a-x}{a+x}$ is a positive quantity; and $\therefore \frac{a^2-x^2}{a^2+x^2}$ is greater than $\frac{a-x}{a+x}$.

LXI. Pages 251—253.

1. (i) $ac : bd :: c^2 : d^2$; if $ac \times d^2 = bd \times c^2$, if $ad = bc$.

(ii) $ab : cd :: a^2 : c^2$; if $ab \times c^2 = cd \times a^2$, if $bc = ad$.

(iii) $a^2 : c^2 = a^2 - b^2 : c^2 - d^2$; if $a^2 \times (c^2 - d^2) = c^2 \times (a^2 - b^2)$;
if $a^2c^2 - a^2d^2 = a^2c^2 - b^2c^2$; if $a^2d^2 = b^2c^2$.

2. Put $\frac{a}{b} = x$, then $\frac{c}{d} = x$; $\therefore a = bx$, $c = dx$;

$$\therefore \frac{2a+3c}{3a+2c} = \frac{2bx+3dx}{3bx+2dx} = \frac{2b+3d}{3b+2d}, \text{ \&c.}$$

3. If $a : b = a+c : b+d$, then $a(b+d) = b(a+c)$;

$$\therefore ad = bc; \quad \therefore c : d = a : b = a+c : b+d.$$

4. Put $\frac{a}{b} = x$, then $\frac{c}{d} = x$; $\therefore a = bx$, $c = dx$.

Hence $$\frac{la+mb}{pa+qb} = \frac{lbx+mb}{pbx+qb} = \frac{lx+m}{px+q} = \frac{ldx+md}{pdx+qd} = \frac{lc+md}{pc+qd}.$$

5. $(3a-5b)(5c+3d) = (3c-5d)(5a+3b)$;

$$\therefore 15ac - 25bc + 9ad - 15bd = 15ac - 25ad + 9bc - 15bd;$$

$$\therefore 34ad = 34bc; \quad \therefore ad = bc; \quad \therefore a : b :: c : d.$$

6. The mean proportional is $\sqrt{a^3b \times ab^3} = a^2b^2$.

7. The mean proportional is $\sqrt{(a+b)^2 \times (a-b)^2} = (a+b)(a-b) = a^2 - b^2$.

8. If $a : a^2 :: a^2 : x$; then $x = \dfrac{a^2 \times a^2}{a} = a^3$.

If $(a-b)^2 : (a^2-b^2) :: (a^2-b^2) : x$; then $x = (a^2-b^2)^2 \div (a-b)^2 = (a+b)^2$.

9. If $a : b = c : d$; then $ad = bc$.

Also $a^2+c^2 : ab+cd :: ab+cd : b^2+d^2$; if $(a^2+c^2)(b^2+d^2) = (ab+cd)^2$;

i.e., if $a^2b^2 + a^2d^2 + b^2c^2 + c^2d^2 = a^2b^2 + c^2d^2 + 2abcd$;

i.e., if $(ad-bc)^2 = 0$.

10. Put $\dfrac{a}{b} = x$; then $\dfrac{c}{d} = x$; $\therefore\ a = bx,\ c = dx$.

Thus (i) $\dfrac{a}{a+c} = \dfrac{bx}{bx+dx} = \dfrac{b}{b+d}$,

also
$$\frac{a+b}{a+b+c+d} = \frac{bx+b}{bx+b+dx+d} = \frac{b(x+1)}{(b+d)(x+1)} = \frac{b}{b+d};$$

$$\therefore \frac{a}{a+c} = \frac{a+b}{a+b+c+d}.$$

(ii) $\dfrac{a^2+ab+b^2}{a^2-ab+b^2} = \dfrac{b^2x^2+b^2x+b^2}{b^2x^2-b^2x+b^2} = \dfrac{x^2+x+1}{x^2-x+1}$;

also
$$\frac{c^2+cd+d^2}{c^2-cd+d^2} = \frac{d^2x^2+d^2x+d^2}{d^2x^2-d^2x+d^2} = \frac{x^2+x+1}{x^2-x+1};$$

$$\therefore \frac{a^2+ab+b^2}{a^2-ab+b^2} = \frac{c^2+cd+d^2}{c^2-cd+d^2}.$$

(iii) $\dfrac{a+b}{c+d} = \dfrac{bx+b}{dx+d} = \dfrac{b}{d}$;

$$\frac{\sqrt{a^2+b^2}}{\sqrt{c^2+d^2}} = \sqrt{\frac{a^2+b^2}{c^2+d^2}} = \sqrt{\frac{b^2x^2+b^2}{d^2x^2+d^2}} = \sqrt{\frac{b^2}{d^2}} = \frac{b}{d};$$

$$\therefore \frac{a+b}{c+d} = \frac{\sqrt{a^2+b^2}}{\sqrt{c^2+d^2}}.$$

(iv) As in (iii) $\dfrac{\sqrt{a^2+b^2}}{\sqrt{c^2+d^2}} = \dfrac{b}{d}$;

also
$$\frac{\sqrt[3]{a^3+b^3}}{\sqrt[3]{c^3+d^3}} = \sqrt[3]{\frac{a^3+b^3}{c^3+d^3}} = \sqrt[3]{\frac{b^3x^3+b^3}{d^3x^3+d^3}} = \sqrt[3]{\frac{b^3}{d^3}} = \frac{b}{d};$$

$$\therefore \frac{\sqrt{a^2+b^2}}{\sqrt{c^2+d^2}} = \frac{\sqrt[3]{a^3+b^3}}{\sqrt[3]{c^3+d^2}}.$$

(v) $\dfrac{a^2c+ac^2}{b^2d+bd^2} = \dfrac{b^2dx^3+bd^2x^3}{b^2d+bd^2} = x^3$; $\dfrac{(a+c)^3}{(b+d)^3} = \dfrac{(bx+dx)^3}{(b+d)^3} = x^3$;

$$\therefore \frac{a^2c+ac^2}{b^2d+bd^2} = \frac{(a+c)^3}{(b+d)^3}.$$

(vi) $$\frac{\sqrt[n]{a^n+b^n}}{\sqrt[n]{c^n+d^n}} = \sqrt[n]{\frac{a^n+b^n}{c^n+d^n}} = \sqrt[n]{\frac{b^nx^n+b^n}{d^nx^n+d^n}} = \sqrt[n]{\frac{b^n}{d^n}} = \frac{b}{d};$$

$$\frac{\sqrt[r]{a^r-b^r}}{\sqrt[r]{c^r-d^r}} = \sqrt[r]{\frac{a^r-b^r}{c^r-d^r}} = \sqrt[r]{\frac{b^rx^r-b^r}{d^rx^r-d^r}} = \sqrt[r]{\frac{b^r}{d^r}} = \frac{b}{d};$$

$$\therefore \frac{\sqrt[n]{a^n+b^n}}{\sqrt[n]{c^n+d^n}} = \frac{\sqrt[r]{a^r-b^r}}{\sqrt[r]{c^r-d^r}}.$$

11. Put $\frac{x}{a}=\lambda$; then $\frac{y}{b}=\lambda$, and $\frac{z}{c}=\lambda$; thus $x=a\lambda$, $y=b\lambda$; $z=c\lambda$.

Then (i) $\frac{x+y}{a+b} = \frac{a\lambda+b\lambda}{a+b} = \lambda = \frac{z+x}{c+a} = \frac{y+z}{b+c}$ similarly.

$$\text{(ii)}\quad (a^2+b^2+c^2)(x^2+y^2+z^2) = (a^2+b^2+c^2)(a^2\lambda^2+b^2\lambda^2+c^2\lambda^2)$$
$$= (a^2+b^2+c^2)^2\lambda^2;$$

also $$(ax+by+cz)^2 = (a^2\lambda+b^2\lambda+c^2\lambda)^2 = (a^2+b^2+c^2)^2\lambda^2, \text{ \&c.}$$

12. By Theorem I, Art. 137, each fraction

$$= \frac{a(bz-cy)+b(cx-az)+c(ay-bx)}{a^2+b^2+c^2} = 0.$$

Hence $bz-cy=0=cx-az=ay-bx$; $\therefore \frac{z}{c}=\frac{y}{b}=\frac{x}{a}$.

13. $\left(\frac{a}{b}\right)^2 - 2\frac{ac}{bd} + \left(\frac{c}{d}\right)^2 = 0$; $\therefore \left(\frac{a}{b}-\frac{c}{d}\right)^2 = 0$.

14. Since $lx(ny-mz) = my(lz-nx) = nz(mx-ly)$;

$$\therefore \frac{x(ny-mz)}{\frac{1}{l}} = \frac{y(lz-nx)}{\frac{1}{m}} = \frac{z(mx-ly)}{\frac{1}{n}}$$

$$= \frac{x(ny-mz)+y(lz-nx)+z(mx-ly)}{\frac{1}{l}+\frac{1}{m}+\frac{1}{n}} = \frac{0}{\frac{1}{l}+\frac{1}{m}+\frac{1}{n}}.$$

Hence $$\frac{1}{l}+\frac{1}{m}+\frac{1}{n}=0; \quad \therefore mn+nl+lm=0.$$

Again $$\frac{l(ny-mz)}{\frac{1}{x}} = \frac{m(lz-nx)}{\frac{1}{y}} = \frac{n(mx-ly)}{\frac{1}{z}}$$

$$= \frac{l(ny-mz)+m(lz-nx)+n(mx-ly)}{\frac{1}{x}+\frac{1}{y}+\frac{1}{z}} = \frac{0}{\frac{1}{x}+\frac{1}{y}+\frac{1}{z}}.$$

Hence $$\frac{1}{x}+\frac{1}{y}+\frac{1}{z}=0; \quad \therefore yz+zx+xy=0.$$

15. Put $$\frac{x}{b+c-a}=\lambda=\&c.$$

Then $$x=\lambda\,(b+c-a),\ \ y=\lambda\,(c+a-b),\ \ z=\lambda\,(a+b-c);$$

$$\therefore\ (b-c)\,x+(c-a)\,y+(a-b)\,z$$
$$=\lambda\,\{(b-c)\,(b+c-a)+(c-a)\,(c+a-b)+(a-b)\,(a+b-c)\}=0.$$

16. Put $\lambda=\frac{a}{b}=\frac{c}{d}=\frac{e}{f}$; then $a=b\lambda,\ c=d\lambda,\ e=f\lambda$;

$$\therefore\ \frac{a^3+a^2c+ace}{b^3+b^2d+bdf}=\frac{b^3\lambda^3+b^2d\lambda^3+bdf\lambda^3}{b^3+b^2d+bdf}=\lambda^3;$$
$$\frac{ace+ac^2+c^3}{bdf+bd^2+d^3}=\frac{bdf\lambda^3+bd^2\lambda^3+d^3\lambda^3}{bdf+bd^2+d^3}=\lambda^3;\ \ \therefore\ \&c.$$

17. Put $\lambda=\frac{a}{x}=\frac{b}{y}=\frac{c}{z}$. Then $a=\lambda x,\ b=\lambda y,\ c=\lambda z$.

$$\therefore\ \frac{a^4+a^2b^2+b^4}{x^4+x^2y^2+y^4}=\frac{\lambda^4x^4+\lambda^4x^2y^2+\lambda^4y^4}{x^4+x^2y^2+y^4}=\lambda^4;$$
$$\frac{b^4+b^2c^2+c^4}{y^4+y^2z^2+z^4}=\frac{\lambda^4y^4+\lambda^4y^2z^2+\lambda^4z^4}{y^4+y^2z^2+z^4}=\lambda^4;\ \ \therefore\ \&c.$$

18. We have $a\,(pb-qd)=b\,(pa-qc)$; $\therefore\ apb-qad=apb-qbc$;
$$\therefore\ qad=qbc;\ \ \therefore\ apb+qad=pab+qbc;$$
i.e. $a\,(pb+qd)=b\,(pa+qc)$; $\therefore$ &c.

19. Put $\frac{a}{c}=\lambda$, then $\frac{b}{d}=\lambda$; $\therefore\ a=c\lambda,\ b=d\lambda$.

$$\therefore\ \frac{a^3}{b}+\frac{b^3}{a}=\frac{c^3\lambda^3}{d\lambda}+\frac{d^3\lambda^3}{c\lambda}=\lambda^2\left(\frac{c^3}{d}+\frac{d^3}{c}\right);$$
$$\therefore\ \frac{a^3}{b}+\frac{b^3}{a}:\frac{c^3}{d}+\frac{d^3}{c}=\lambda^2:1=\lambda^2cd:cd=ab:cd.$$

20. $2\,(4a-b)=4a+b$; $\therefore 4a=3b$.

Hence $$7a+3b:7a-3b=7a+4a:7a-4a=11a:3a=11:3.$$

21. $(xy+3)\,(yz+1)-(xz+1)\,(y^2+3)=0$;
$$\therefore\ xy^2z+3yz+xy+3-(xy^2z+y^2+3xz+3)=0;$$
$$\therefore\ 3yz-3xz+xy-y^2=0;\ \textit{i.e.}\ (3z-y)\,(x-y)=0,\ \&c.$$

22. We have $$(a+b-c)\,2d=(c+d+a)\,(a-c);$$
$$\therefore\ 2ad+2bd-2cd=a^2+ad-c^2-cd;$$
$$\therefore\ 2bd=a^2-ad-c^2+cd=(a-c)\,(a+c-d);$$
$$\therefore\ b:a-c::a+c-d:2d.$$

23. We have $(x-z)\,y^2=(y-z)\,x^2$;

$$\therefore\ z\,(x^2-y^2)=x^2y-xy^2;\ \ \therefore\ z\,(x+y)=xy.$$

Hence $\dfrac{x+z}{y+z}=\dfrac{x\,(x+y)+z\,(x+y)}{y\,(x+y)+z\,(x+y)}=\dfrac{x\,(x+y)+xy}{y\,(x+y)+xy}=\dfrac{x^2+2xy}{y^2+2xy}$.

24. $y-z:b-c=z-x:c-a$, if $(b-c)\,(z-x)-(c-a)\,(y-z)=0$,
that is, if $a\,(y-z)+b\,(z-x)-c\,\{z-x+y-z\}=0$,
that is, if $a\,(y-z)+b\,(z-x)+c\,(x-y)=0$,
which is given. Similarly for the other relation.

25. Put $\dfrac{a}{b}=\lambda$, then $\dfrac{c}{d}=\lambda$; $\therefore\ a=b\lambda,\ c=d\lambda$.

$$\therefore\ \frac{a^2+b^2+c^2+d^2}{(a+b)^2+(c+d)^2}=\frac{b^2\lambda^2+b^2+d^2\lambda^2+d^2}{(\lambda b+b)^2+(\lambda d+d)^2}=\frac{(\lambda^2+1)\,(b^2+d^2)}{(\lambda+1)^2\,(b^2+d^2)}=\frac{\lambda^2+1}{(\lambda+1)^2}.$$

Also $\dfrac{(a+c)^2+(b+d)^2}{(a+b+c+d)^2}=\dfrac{(b\lambda+d\lambda)^2+(b+d)^2}{(b\lambda+b+d\lambda+d)^2}=\dfrac{(\lambda^2+1)\,(b+d)^2}{(\lambda+1)^2\,(b+d)^2}=\dfrac{\lambda^2+1}{(\lambda+1)^2}$.

LXII. Pages 257, 258.

1. $A=mB$; $\therefore\ 5=m\times3$; $\therefore\ A=\dfrac{5}{3}B=\dfrac{25}{3}$, when B is 5.

2. $W=m\,.\,\dfrac{1}{P}$; $\therefore\ 4=\dfrac{m}{15}$. $W=\dfrac{60}{P}=\dfrac{60}{12}$, when P is 12.

3. $x=my$ and $y=nz$, where m and n are constants; $\therefore\ xz=\dfrac{m}{n}y^2$.

4. $x^2=my$ and $z^2=ny$; $\therefore\ xz=\sqrt{mn}\,.\,y$.

5. $x=\dfrac{m}{y}$ and $y=\dfrac{n}{z}$; $\therefore\ x=\dfrac{m}{n}z$.

6. $A=mBC$; $4=m\times2\times6$; $\therefore\ m=\dfrac{1}{3}$. Then $A=\dfrac{1}{3}\times2\times9$; $\therefore\ A=6$.

7. $A=m\times B\times\dfrac{1}{C}$. $2=m\times3\times\dfrac{1}{4}$; $\therefore\ m=\dfrac{8}{3}$. Then

$$6=\frac{8}{3}\times B\times\frac{1}{3};\ \ \therefore\ B=\frac{27}{4}.$$

8. $A=m\,.\,r^2$; $314{\cdot}159=m\times100$; $\therefore\ m=3{\cdot}14159$. $A=3{\cdot}14159\times12^2=$&c.

9. $V=m\,.\,r^3$; $4{\cdot}188=m\,.\,1^3$; $\therefore\ V=4{\cdot}188\times3^3=$&c.

10. $V=m\,.\,T$; $64=m\times2$; $\therefore\ m=32$. Then $V=32\times5=160$.

11. $D=m\,.\,T^2$; $144=m\times9$; $\therefore\ m=16$. Then $D=16\times2^2=64$.

12. $A=m\,.\,r^2$. Hence the area of a circle of 5 feet radius
$=m\,.\,5^2=m\,(3^2+4^2)$
$=$area of circle of 3 feet radius$+$area of circle of 4 feet radius.

13. $V=m\times T\times\frac{1}{P}$; $1=m\,.\,\frac{280}{15}$. Hence $V=\frac{15}{280}\,.\,\frac{300}{20}=\frac{45}{56}$.

14. $a^2-b^2=mc^2$; $5^2-3^2=m\times 2^2$; $\therefore\ m=4$. Thus

$$a^2-b^2=4c^2;\ \therefore\ b^2=a^2-4c^2;\ \therefore\ b=\sqrt{(a+2c)(a-2c)}.$$

15. $V=m\,.\,H\,.\,R^2$; $66=m\times 7\times 3^2$; $\therefore\ m=\frac{22}{21}$.

Then
$$V=\frac{22}{21}\times 9\times 14^2=1848.$$

16. $V=m\,.\,r^3=m\,.\,6^3+m\,.\,8^3+m\,.\,10^3=m\,(216+512+1000)$

$$=m\,.\,1728=m\,.\,12^3.\quad \text{Hence } r=12 \text{ inches.}$$

17. Let h be the height of the eye above the sea-level, d the distance of the offing.

Then $d\propto\sqrt{h}$; $\therefore\ d=m\sqrt{h}$, but $d=3$ miles when $h=6$ ft.;

$$\therefore\ 3=m\sqrt{6};\ \therefore\ m=\frac{3}{\sqrt{6}}=\frac{\sqrt{3}}{\sqrt{2}}.$$

Thus $d=\frac{\sqrt{3}}{\sqrt{2}}\sqrt{h}$; $\therefore$ when h is 150 ft., $d=\frac{\sqrt{3}}{\sqrt{2}}\sqrt{150}=\sqrt{225}=15$ miles.

N.B. We may express d and h in terms of different units, but the unit for each must be the same throughout.

MISCELLANEOUS EXAMPLES. V.

A. 1. $\frac{2a}{3}=1$; $\therefore\ a=\frac{3}{2}$; similarly $b=\frac{4}{3}$, $c=2$.

$$\therefore\ (a-b)^2+(b-c)^2+(c-a)^2=\left(\frac{3}{2}-\frac{4}{3}\right)^2+\left(\frac{4}{3}-2\right)^2+\left(2-\frac{3}{2}\right)^2$$

$$=\left(\frac{1}{6}\right)^2+\left(-\frac{2}{3}\right)^2+\left(\frac{1}{2}\right)^2=\&c.$$

2.
$$\begin{array}{l}
4a^3-2ab^2+3b^3\ )\ 4a^5-12a^4b+2a^3b^2+9a^2b^3-11ab^4+3b^5\ (\ a^2-3ab+b^2\\
\qquad\qquad\qquad\quad\ \ 4a^5\qquad\quad\ \ -2a^3b^2+3a^2b^3\\
\qquad\qquad\qquad\qquad\ \ -12a^4b+4a^3b^2+6a^2b^3-11ab^4+3b^5\\
\qquad\qquad\qquad\qquad\ \ -12a^4b\qquad\quad\ \ +6a^2b^3-\ \ 9ab^4\\
\qquad\qquad\qquad\qquad\qquad\qquad\ \ 4a^3b^2\qquad\qquad\ -\ \ 2ab^4+3b^5\\
\qquad\qquad\qquad\qquad\qquad\qquad\ \ 4a^3b^2\qquad\qquad\ -\ \ 2ab^4+3b^5
\end{array}$$

3. $$\begin{array}{l}
1-x-x^3+x^5\;)\;1-x^4-x^6-x^7\;(\;1+x+x^2\\
\qquad\qquad\qquad\;\;\underline{1-x\;-x^3+x^5\qquad\qquad\quad}\\
\qquad\qquad\qquad\qquad\;x\;+x^3-x^4-x^5-\;x^6-\;x^7\\
\qquad\qquad\qquad\qquad\;\underline{x\;-x^2-x^4\qquad+\;x^6\qquad}\\
\qquad\qquad\qquad\qquad\qquad x^2+x^3-x^5-2x^6-\;x^7\\
\qquad\qquad\qquad\qquad\qquad \underline{x^2-x^3-x^5\qquad\;+\;x^7}\\
\qquad\qquad\qquad\qquad\;2x^3\;)\;2x^3\qquad-2x^6-2x^7\\
\qquad\qquad\qquad\qquad\qquad\quad\;1-\qquad\;x^3-\;x^4\;)\;1-x-x^3+x^5\;(\;1-x\\
\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\;\;\underline{1\qquad-x^3-x^4}\\
\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\quad\;-x+x^4+x^5\\
\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\quad\;-x+x^4+x^5
\end{array}$$

4. Multiply numerator and denominator by b^2; thus

$$\frac{a^2b^2-a^2-b^2+1}{ab^2+ab+b^2+b}=\frac{(a^2-1)(b^2-1)}{(a+1)(b+1)b}=\frac{(a-1)(b-1)}{b}.$$

5. Let a duck cost x shillings, and a goose y shillings.

Then $$15x+12y=105;\ \therefore\ 5x+4y=35.$$

Also for 18*s.* we can buy $\frac{18}{x}$ ducks;

and for 20*s.* ,, ,, ,, $\frac{20}{y}$ geese;

$$\therefore\ \frac{18}{x}-2=\frac{20}{y}.$$

Hence $$\frac{90}{5x}-2=\frac{20}{y};\ \therefore\ \frac{90}{35-4y}-2=\frac{20}{y};$$

$$\therefore\ 90y-2y(35-4y)=20(35-4y);$$

$$\therefore\ 2y^2+25y-175=0,\ i.e.\ (2y+35)(y-5)=0$$

Hence $y=5$, and then $x=3$.

6. Let α_1, β_1 be the roots of the first equation; α_2, β_2 of the second.

Then (§ 150) $$\alpha_1+\beta_1=p,\ \alpha_1\beta_1=q;$$

$$\therefore\ (\alpha_1-\beta_1)^2=(\alpha_1+\beta_1)^2-4\alpha_1\beta_1=p^2-4q.$$

Similarly $$\alpha_2+\beta_2=3p,\ \alpha_2\beta_2=2p^2+q;$$

$$\therefore\ (\alpha_2-\beta_2)^2=(\alpha_2+\beta_2)^2-4\alpha_2\beta_2=9p^2-4(2p^2+q)=p^2-4q.$$

7. Let x be such that $3+x:4+x>19:21$; then $21(3+x)>19(4+x)$, whence $2x>13$; $\therefore\ x>6\frac{1}{2}$. Thus the least integer required is 7.

8. We have $$a+b:b+c=b+c:c+a;$$

$$\therefore\ (a+b)(c+a)=(b+c)^2;\ i.e.\ a^2+ab+ac+bc=b^2+2bc+c^2;$$

$$\therefore\ ab+ac-b^2-bc=c^2-a^2;\ i.e.\ (b+c)(a-b)=(c+a)(c-a);$$

$$\therefore\ b+c:c+a=c-a:a-b.$$

B. 1. $\sqrt{1-x^2}=\sqrt{1-\frac{16}{25}}=\sqrt{\frac{9}{25}}=\frac{3}{5}$,

and $$\sqrt{1-y^2}=\sqrt{1-\frac{9}{25}}=\sqrt{\frac{16}{25}}=\frac{4}{5}.$$

$$\therefore\ (xy+\sqrt{1-x^2}\sqrt{1-y^2})\div(x\sqrt{1-y^2}-y\sqrt{1-x^2})$$
$$=\left(\frac{12}{25}+\frac{12}{25}\right)\div\left(\frac{16}{25}-\frac{9}{25}\right)=\frac{24}{25}\div\frac{7}{25}=\frac{24}{7}.$$

2.
$$\begin{array}{l}
x^2+(a-1)\,x+(a+1)\\
(a-1)\,x-(a^2+a+1)\\
\hline
(a-1)\,x^3+(a^2-2a+1)\,x^2+(a^2-1)\,x\\
\qquad\qquad -(a^2+\ a+1)\,x^2-(a^3-1)\ \ x-(a^3+2a^2+2a+1)\\
\hline
(a-1)\,x^3\qquad\qquad\qquad -3ax^2+(a^2-a^3)\,x-(a^3+2a^2+2a+1)
\end{array}$$

3. (i) $x\,(x^2-13xy+42y^2)=x\,(x-6y)\,(x-7y)$.

(ii) $\{(a+2b+3c)-2\,(a+b-c)\}\,\{(a+2b+3c)+2\,(a+b-c)\}=$&c.

(iii) $(x-2y)^2-9=\{(x-2y)+3\}\,\{(x-2y)-3\}=$&c.

4. $$\frac{x}{x-2y}+\frac{3x}{x+2y}+\frac{-4xy}{x^2-4y^2}=\frac{x\,(x+2y)+3x\,(x-2y)-4xy}{(x-2y)\,(x+2y)}$$
$$=\frac{4x^2-8xy}{(x-2y)\,(x+2y)}=\frac{4x\,(x-2y)}{(x-2y)\,(x+2y)}=\text{\&c.}$$

5. (i) Multiply by $x\,(x+2)\,(x+1)$; then
$$10\,(x+2)\,(x+1)-3x\,(x+1)=10x\,(x+2);$$
whence $$-3x^2+7x+20=0;\ \textit{i.e.}\ -(3x+5)\,(x-4)=0,\ \text{\&c.}$$

(ii) $x+y+\sqrt{x+y}-12=0$, *i.e.* $(\sqrt{x+y}+4)\,(\sqrt{x+y}-3)=0$;
$$\therefore\ \sqrt{x+y}=-4\ \text{or}\ 3;\ \therefore\ x+y=16\ \text{or}\ 9.$$

Then, from $x+y=16$ and $x^2-y^2=21$, or $(x+y)\,(x-y)=21$; we have $x-y=\frac{21}{16}$. Hence $x=8+\frac{21}{32}$, $y=8-\frac{21}{32}$. And, from $x+y=9$ and $x^2-y^2=21$, we have $x-y=\frac{7}{3}$, &c.

6. From $x+\frac{1}{y}=1$, we have $\frac{1}{y}=1-x$. Hence $y+\frac{1}{z}=1$ becomes
$$\frac{1}{1-x}+\frac{1}{z}=1;$$
$$\therefore\ z+1-x=z\,(1-x);\ \therefore\ 1+xz=x;\ \therefore\ \frac{1}{x}+z=1.$$

Again, since $z=1-\frac{1}{x}$, $xyz=xy\left(1-\frac{1}{x}\right)=y\,(x-1)=y\left(-\frac{1}{y}\right)$
$$\left[\text{from } x+\frac{1}{y}=1\right]=-1.$$

7.
$$\begin{array}{r l}
 & 9x^6-12x^4y^2+30x^3y^3+4x^2y^4-20xy^5+25y^6\ (\ 3x^3-2xy^2+5y^3 \\
(3x^3)^2= & 9x^6 \\
(3x^3-2xy^2)^2= & \overline{9x^6-12x^4y^2+\ \ 4x^2y^4} \\
(3x^3-2xy^2+5y^3)^2= & \overline{9x^6-12x^4y^2+30x^3y^3+4x^2y^4-20xy^5+25y^6}
\end{array}$$

8. Let x and y be the numbers; then

$$x+y : x-y : x^2+y^2 :: 3 : 1 : 15.$$

From $x+y : x-y=3 : 1$, $x+y=3(x-y)$; $\therefore\ x=2y$.

Then, from $x-y : x^2+y^2=1 : 15$, $15(x-y)=x^2+y^2$.

Hence $15y=4y^2+y^2$; $\therefore$ (rejecting $y=0=x$) we have $y=3$, and then $x=6$.

9. Let $\frac{a}{b}=x$, then $\frac{c}{d}=x$; $\therefore\ a=bx,\ c=dx$.

$$\therefore\ \frac{la+mb}{lc+md}=\frac{lbx+mb}{ldx+md}=\frac{(lx+m)\,b}{(lx+m)\,d}=\frac{b}{d}.$$

Also
$$\frac{\sqrt{a^3+b^2d}}{\sqrt{ac^2+d^3}}=\sqrt{\frac{a^3+b^2d}{ac^2+d^3}}=\sqrt{\frac{b^3x^3+b^2d}{bd^2x^3+d^3}}$$
$$=\sqrt{\frac{b^2(bx^3+d)}{d^2(bx^3+d)}}=\sqrt{\frac{b^2}{d^2}}=\frac{b}{d};\ \therefore\ \&c.$$

C. 1. $3\{b+2(a-c)-5(a-b)\}-\{a-2(b-c)\}=3\{-3a+6b-2c\}$
$-\{a-2b+2c\}=-9a+18b-6c-a+2b-2c=-10a+20b-8c.$

2.
$$\left\{\left(a+\frac{b^2}{a}\right)+\left(b+\frac{a^2}{b}\right)\right\}\left\{\left(a+\frac{b^2}{a}\right)-\left(b+\frac{a^2}{b}\right)\right\}$$
$$=\left(a+\frac{b^2}{a}\right)^2-\left(b+\frac{a^2}{b}\right)^2=\left(a^2+2b^2+\frac{b^4}{a^2}\right)-\left(b^2+2a^2+\frac{a^4}{b^2}\right)$$
$$=b^2-a^2+\frac{b^4}{a^2}-\frac{a^4}{b^2}.$$

3. (i)
$$\frac{a^2+b^2}{9(a^2-b^2)}\times 27(a-b)=\frac{27(a-b)(a^2+b^2)}{9(a-b)(a+b)}=\frac{3(a^2+b^2)}{a+b}.$$

(ii)
$$\frac{9a^2-16b^2}{a+b}\div\frac{3a-4b}{a^2-b^2}=\frac{(3a-4b)(3a+4b)}{a+b}\times\frac{(a-b)(a+b)}{3a-4b}$$
$$=(3a+4b)(a-b)=\&c.$$

4. Substitute for a, b, c on the left. **Or thus:**
$a-b=(y+z)-(z+x)=y-x$; and similarly $b-c=z-y$, and $c-a=x-z$.

Then $2(a^2+b^2+c^2-bc-ca-ab)=(b-c)^2+(c-a)^2+(a-b)^2$
$$=(z-y)^2+(x-z)^2+(y-x)^2=2(x^2+y^2+z^2-yz-zx-xy).$$

5.

$$\begin{array}{ll}
 & a^{\frac{4}{3}}+4ay^{\frac{1}{5}}+10a^{\frac{2}{3}}y^{\frac{2}{5}}+12a^{\frac{1}{3}}y^{\frac{3}{5}}+9y^{\frac{4}{5}}\ (\ a^{\frac{2}{3}}+2a^{\frac{1}{3}}y^{\frac{1}{5}}+3y^{\frac{2}{5}} \\
(a^{\frac{2}{3}})^2 & =a^{\frac{4}{3}} \\ \hline
(a^{\frac{2}{3}}+2a^{\frac{1}{3}}y^{\frac{1}{5}})^2 & =a^{\frac{4}{3}}+4ay^{\frac{1}{5}}+\ 4a^{\frac{2}{3}}y^{\frac{2}{5}} \\ \hline
(a^{\frac{2}{3}}+2a^{\frac{1}{3}}y^{\frac{1}{5}}+3y^{\frac{2}{5}})^2 & =a^{\frac{4}{3}}+4ay^{\frac{1}{5}}+10a^{\frac{2}{3}}y^{\frac{2}{5}}+12a^{\frac{1}{3}}y^{\frac{3}{5}}+9y^{\frac{4}{5}}
\end{array}$$

6. $(x-1)(x-2)=(1+\sqrt{2})\sqrt{2}=\sqrt{2}+2=x.$

7. Let A and B travel at x and $x+2$ miles per hour respectively, and let d be the distance in miles.

Then in 7 hours they between them travel the whole distance;

$$\therefore\ 7x+7(x+2)=d.$$

Also in 9 hours they would travel the whole distance between them at $\frac{x}{2}$ and $x+3$ miles per hour respectively;

$$\therefore\ \frac{x}{2}\times 9+9(x+3)=d.$$

Hence $\qquad 14x=d-14$ and $27x=2d-54$;

$$\therefore\ 27(d-14)=14(2d-54),\ \text{whence}\ d=378.$$

8. We have $\dfrac{y+z}{\frac{1}{a}}=\dfrac{z+x}{\frac{1}{b}}=\dfrac{x+y}{\frac{1}{c}}$;

$$\therefore\ [\text{Art. } 137]\ \text{each}=\frac{y-z}{\frac{1}{c}-\frac{1}{b}}=\frac{z-x}{\frac{1}{a}-\frac{1}{c}}=\frac{x-y}{\frac{1}{b}-\frac{1}{a}}=\frac{abc(y-z)}{a(b-c)}$$

$$=\frac{abc(z-x)}{b(c-a)}=\frac{abc(x-y)}{c(a-b)}.$$

Hence $\qquad \dfrac{y-z}{a(b-c)}=\dfrac{z-x}{b(c-a)}=\dfrac{x-y}{c(a-b)}.$

D. 1. $\left(\dfrac{a^3-b^3}{a-b}\right)^2-\left(\dfrac{a^3+b^3}{a+b}\right)^2=(a^2+ab+b^2)^2-(a^2-ab+b^2)^2$

$$=\{(a^2+ab+b^2)-(a^2-ab+b^2)\}\{(a^2+ab+b^2)+(a^2-ab+b^2)\}$$

$$=\{2ab\}\{2(a^2+b^2)\}=\&\text{c}.$$

2.

$$\begin{array}{lr}
x^2-(a+b)x+ab\ ) & x^4-2bx^3+(-a^2+b^2)x^2+2a^2bx-a^2b^2\ (\ x^2+(a-b)x-ab \\
 & x^4-(a+b)x^3+abx^2 \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \\ \hline
 & (a-b)x^3+(-a^2-ab+b^2)x^2+2a^2bx-a^2b^2 \qquad\qquad\quad \\
 & (a-b)x^3-(\ a^2\qquad\ -b^2)x^2+(a^2b-ab^2)x \qquad\qquad\qquad\quad \\ \hline
 & -\ abx^2+(a^2b+ab^2)x-a^2b^2 \\
 & -\ abx^2+(a^2b+ab^2)x-a^2b^2
\end{array}$$

3. $x^2-7x+12=(x-3)(x-4)$, $3x^2-6x-9=3(x-3)(x+1)$, and $$2x^3-6x^2-8x=2x(x-4)(x+1).$$

Thus L. C. M. is $6x(x-3)(x-4)(x+1)$.

4. $$\frac{\left\{\dfrac{y}{x}-\dfrac{y}{y+z}\right\}xz(y+z)}{\left\{\dfrac{1}{xz}+\dfrac{1}{z(y+z)}\right\}xz(y+z)}\times\frac{2yz+y^2+z^2-x^2}{2yz}$$

$$=\frac{yz(y+z)-xyz}{(y+z)+x}\times\frac{(y+z)^2-x^2}{2yz}=\frac{yz(y+z-x)}{y+z+x}\times\frac{(y+z+x)(y+z-x)}{2yz}$$

$$=\tfrac{1}{2}(y+z-x)^2.$$

5. (i) Multiply by 6; then $(15x+21)-(4x-14)=18x+84$; whence $x=-7$.

(ii) $14(y^2-xy)=15(x^2+xy)$; whence $14y^2-29xy-15x^2=0$;

$$i.e.\ (2y-5x)(7y+3x)=0;\ \therefore\ y=\frac{5x}{2},\ \text{or}\ -\frac{3x}{7}.$$

Then, if $y=\dfrac{5x}{2}$; $x^2+\dfrac{5x^2}{2}=14$; whence $x^2=4$;

$$\therefore\ x=\pm 2;\ \text{and}\ y=\frac{5x}{2}=\pm 5.$$

And, if $y=-\dfrac{3x}{7}$; $\therefore\ x^2-\dfrac{3x^2}{7}=14$; whence $x^2=\dfrac{49}{2}$;

$$\therefore\ x=\pm\frac{7}{\sqrt{2}}=\pm\frac{7}{2}\sqrt{2};\ \text{and}\ \therefore\ y=-\frac{3x}{7}=\mp\frac{3}{2}\sqrt{2}.$$

6. Let the wheels make x and $x-64$ revolutions respectively in a mile.

Then $$10x+10(x-64)=7040,\ \text{whence}\ x=384.$$

Hence circumference of fore-wheel in yards × 384 = 1760, and circumference of hind-wheel in yards × 320 = 1760.

Hence the circumferences are $4\frac{7}{12}$ yds. and $5\frac{1}{2}$ yds. respectively.

7. $$\begin{array}{l} 2x^{\frac{1}{2}}-15y^{\frac{1}{2}}\ )\ 2x^{\frac{3}{2}}-17xy^{\frac{1}{2}}+17x^{\frac{1}{2}}y-15y^{\frac{3}{2}}\ (\ x-x^{\frac{1}{2}}y^{\frac{1}{2}}+y \\ \qquad\qquad\quad 2x^{\frac{3}{2}}-15xy^{\frac{1}{2}} \\ \hline \qquad\qquad\qquad -\ 2xy^{\frac{1}{2}}+17x^{\frac{1}{2}}y-15y^{\frac{3}{2}} \\ \qquad\qquad\qquad -\ 2xy^{\frac{1}{2}}+15x^{\frac{1}{2}}y \\ \hline \qquad\qquad\qquad\qquad\qquad 2x^{\frac{1}{2}}y-15y^{\frac{3}{2}} \\ \qquad\qquad\qquad\qquad\qquad 2x^{\frac{1}{2}}y-15y^{\frac{3}{2}} \end{array}$$

8. We have $\dfrac{a}{\alpha}=\dfrac{b}{\beta}=\dfrac{c}{\gamma}=\lambda$ (say); $\therefore\ a=\lambda\alpha,\ b=\lambda\beta,\ c=\lambda\gamma.$

$$\therefore\ a+b+c=\lambda\alpha+\lambda\beta+\lambda\gamma=\lambda(\alpha+\beta+\gamma);\ \therefore\ \frac{a+b+c}{\alpha+\beta+\gamma}=\lambda;\ \therefore\ \&c.$$

E. 1. $2\left\{x-2a-\frac{2}{3}x+\frac{2}{3}a+a\right\}-\frac{1}{3}\{x-a-a+b+y-b\}$

$$=2\left\{\frac{1}{3}x-\frac{1}{3}a\right\}-\frac{1}{3}\{x-2a+y\}$$

$$=\frac{2}{3}x-\frac{2}{3}a-\frac{1}{3}x+\frac{2}{3}a-\frac{1}{3}y=\frac{1}{3}x-\frac{1}{3}y.$$

2.
$$\begin{array}{l|l|l} ax+2(b-c)y & 2a^2x^2+abxy-2(3b-4c)(b-c)y^2 & 2ax-(3b-4c)y \\ & \underline{2a^2x^2+4a(b-c)xy} & \\ & \quad(-3ab+4ac)xy-2(3b-4c)(b-c)y^2 & \\ & \quad(-3ab+4ac)xy-2(3b-4c)(b-c)y^2 & \end{array}$$

3. (i) $4a^2b^2(b^2-4a^2)=$&c.

4. $\left(\frac{1}{y-z}+\frac{1}{z-x}+\frac{1}{x-y}\right)^2$

$$=\frac{1}{(y-z)^2}+\frac{1}{(z-x)^2}+\frac{1}{(x-y)^2}+\frac{2}{(y-z)(z-x)}+\frac{2}{(y-z)(x-y)}+\frac{2}{(z-x)(x-y)}$$

$$=\frac{1}{(y-z)^2}+\frac{1}{(z-x)^2}+\frac{1}{(x-y)^2}+\frac{2(x-y)+2(z-x)+2(y-z)}{(y-z)(z-x)(x-y)}$$

$$=\frac{1}{(y-z)^2}+\frac{1}{(z-x)^2}+\frac{1}{(x-y)^2}.$$

5. Let x be the number of articles; then each article cost £$\frac{21}{x}$. If he had bought $(x+6)$ for £21, each would have cost £$\frac{21}{x+6}$.

Thus, from the question, £$\frac{21}{x+6}$ is less than £$\frac{21}{x}$ by £$\frac{1}{16}$;

$$\therefore \frac{21}{x+6}=\frac{21}{x}-\frac{1}{16};\ \therefore x^2+6x-2016=0;\ i.e.\ (x+48)(x-42)=0.$$

Hence $x=42$, the negative value being inadmissible.

6. $\{\surd(x-a)+(\surd a-\surd x)\}\{\surd(x-a)-(\surd a-\surd x)\}=\{(x-a)-(\surd a-\surd x)^2\}$

$$=(x-a)-(a-2\sqrt{ax}+x)=2\sqrt{ax}-2a.$$

7. The mean proportional is $\sqrt{\left(x^2-\frac{1}{y^2}\right)\left(y^2-\frac{1}{x^2}\right)}$

$$=\sqrt{\left(x^2y^2-2+\frac{1}{x^2y^2}\right)}=xy-\frac{1}{xy}.$$

8. Put $x=\frac{a}{b}=\frac{c}{d}$; $\therefore a=bx,\ c=dx.$

$$\therefore \frac{a^2+b^2}{c^2+d^2}=\frac{b^2x^2+b^2}{d^2x^2+d^2}=\frac{b^2(x^2+1)}{d^2(x^2+1)}=\frac{b^2}{d^2}.$$

Also $\frac{(a+b)^2}{(c+d)^2}=\frac{(bx+b)^2}{(dx+d)^2}=\frac{b^2(x+1)^2}{d^2(x+1)^2}=\frac{b^2}{d^2}$; $\therefore$ &c.

F. 1. $\{a-(b-c+d)\}\{a+(b-c+d)\}=a^2-(b-c+d)^2$

$$=a^2-\{b^2+c^2+d^2+2b(-c)+2bd+2(-c)d\}$$

$$=a^2-\{b^2+c^2+d^2-2bc+2bd-2cd\}=\&c.$$

2. We proceed to arrange in descending powers of a [Art. 87]. Thus

$$a^3+b^3+c^3+3(b+c)\{a^2+a(b+c)+bc\}$$

$$=a^3+3a^2(b+c)+3a(b+c)^2+b^3+c^3+3bc(b+c).$$

The division could now be performed in the ordinary manner; but noticing that $b^3+c^3+3bc(b+c)=(b+c)^3$, we have

$$a^3+3a^2(b+c)+3a(b+c)^2+(b+c)^3 \text{ which is } (a+b+c)^3.$$

And $$(a+b+c)^3\div(a+b+c)=(a+b+c)^2.$$

3.
$$\begin{array}{rl} x^3-19x+30\;)\; & 5x^3-19x^2+36\;(\;5 \\ & 5x^3-95x+150 \\ \hline -19\;)\; & -19x^2+95x-114 \\ \hline & x^2-5x+6\;)\;x^3-19x+30\;(\;x+5 \\ & \qquad\qquad\qquad x^3-5x^2+6x \\ \hline & \qquad\qquad\qquad 5x^2-25x+30 \\ & \qquad\qquad\qquad 5x^2-25x+30 \end{array}$$

Thus H. C. F. is x^2-5x+6.

Since x^2-5x+6 is a factor of both expressions, they will both vanish if $x^2-5x+6=0$, *i.e.* if $(x-2)(x-3)=0$; *i.e.* if $x=2$ or if $x=3$.

4. $$\frac{(a+b)^2}{(a-b)^2}-\frac{(a-b)^2}{(a+b)^2}=\frac{(a+b)^4-(a-b)^4}{(a-b)^2(a+b)^2}$$

$$=\frac{\{(a+b)^2-(a-b)^2\}\{(a+b)^2+(a-b)^2\}}{\{(a-b)(a+b)\}^2}$$

$$=\frac{\{(a^2+2ab+b^2)-(a^2-2ab+b^2)\}\{(a^2+2ab+b^2)+(a^2-2ab+b^2)\}}{\{a^2-b^2\}^2}$$

$$=\frac{\{4ab\}\{2(a^2+b^2)\}}{(a^2-b^2)^2}=\&c.$$

5. (i) $\dfrac{6x-7}{3(3x+2)}-\dfrac{5(x-1)}{4(3x+2)}=\dfrac{1}{2}$; $\therefore\ 4(6x-7)-15(x-1)=6(3x+2)$, &c.

(ii) $4x=3-6y$; $\therefore\ x=\dfrac{3-6y}{4}$.

Hence $4\left(\dfrac{3-6y}{4}\right)^2+9y\left(\dfrac{3-6y}{4}\right)+9y^2=11$;

$$\therefore\ 18y^2-9y-35=0,\ \textit{i.e.}\ (3y-5)(6y+7)=0.$$

Hence $y=\dfrac{5}{3}$ or $-\dfrac{7}{6}$. Then $x=\dfrac{3-6y}{4}=-\dfrac{7}{4}$ or $\dfrac{5}{2}$.

6. (i) Multiply both numerator and denominator by $6\sqrt{6}$.

Thus we have $\dfrac{12\sqrt{2}+18\sqrt{3}}{3\sqrt{2}-2\sqrt{3}}=\dfrac{(12\sqrt{2}+18\sqrt{3})(3\sqrt{2}+2\sqrt{3})}{(3\sqrt{2}-2\sqrt{3})(3\sqrt{2}+2\sqrt{3})}$

$$=\frac{72+24\sqrt{6}+54\sqrt{6}+108}{18-12}=30+13\sqrt{6}.$$

(ii) $x^{\frac{1}{2}}y^{\frac{1}{3}}\times(x^{-\frac{2}{3}}y^{-1})\div\dfrac{1}{x^{\frac{1}{6}}y^{\frac{1}{4}}}$

$$=x^{\frac{1}{2}}y^{\frac{1}{3}}\times x^{-\frac{2}{3}}y^{-1}\times x^{\frac{1}{6}}y^{\frac{1}{4}}=x^{\frac{1}{2}-\frac{2}{3}+\frac{1}{6}}y^{\frac{1}{3}-1+\frac{1}{4}}=x^0y^{-\frac{5}{12}}=y^{-\frac{5}{12}}.$$

7.

$$\begin{array}{ll}
 & x^{12}-6x^{10}+13x^8-14x^6+10x^4-4x^2+1\ (\ x^6-3x^4+2x^2-1 \\
(x^6)^2 & =x^{12} \\
(x^6-3x^4)^2 & =\overline{x^{12}-6x^{10}+\ \ 9x^8} \\
(x^6-3x^4+2x^2)^2 & =\overline{x^{12}-6x^{10}+13x^8-12x^6+\ \ 4x^4} \\
(x^6-3x^4+2x^2-1)^2 & =\overline{x^{12}-6x^{10}+13x^8-14x^6+10x^4-4x^2+1}
\end{array}$$

8. Put $\dfrac{a}{b}=\dfrac{c}{d}=x$; $\therefore\ a=bx,\ c=dx.$

$$\therefore\ \frac{a^3c^2}{b^3d^2}=\frac{b^3d^2x^5}{b^3d^2}=x^5;\ \text{and}\ \frac{a^5+c^5}{b^5+d^5}=\frac{b^5x^5+d^5x^5}{b^5+d^5}=x^5;\ \therefore\ \&\text{c}.$$

LXIII. PAGES 266—268.

1. The 30th term is $a+29d$, where (i) $a=3,\ d=2$; (ii) $a=1,\ d=4$; (iii) $a=12,\ d=-3$; (iv) $a=\dfrac{1}{2},\ d=\dfrac{2}{9}$; (v) $\alpha=a+b,\ \delta=-b$.

2. (i) $a+23d$, where $a=3,\ d=3$. (ii) $a+29d$, where $a=5,\ d=4$.
(iii) $a+9d$, where $a=6,\ d=-1$. (iv) $a+11d$, where $a=14,\ d=32$.
(v) $a+13d$, where $a=6,\ d=2\frac{2}{3}$. (vi) $a+24d$, where $a=\dfrac{1}{3},\ d=-\dfrac{5}{6}$.

3. $a+9d=6,\ a+5d=10$. Hence $a=15$.

4. $a+11d=15,\ a+19d=25$. Hence $d=\dfrac{5}{4}$.

5. $a+6d=5,\ a+11d=30$. Hence $d=5$.

6. $a+2d=40,\ a+12d=25$. Hence $a=43$.

7. $a=7,\ a+2d=13$; $\therefore\ d=3$. Hence $a+9d=7+27=34$.

8. $a=20$ and $a+5d=10$; $\therefore\ d=-2$. Then $a+11d=20-22=-2$.

9. $a+2d=10,\ a+13d=54$; hence $a=2,\ d=4$. Then
$$a+19d=2+19\times4=78.$$

10. $a+6d=5,\ a+4d=7$; $\therefore\ d=-1$. Hence $a+11d=0$.

11. The nth term is $5+(n-1)\,3$. This is equal to 65 if $3n+2=65$; $\therefore n=21$.

12. $\frac{7}{6}+(n-1)\left(\frac{4}{3}-\frac{7}{6}\right)=18$; whence $n=102$.

13. $9+(n-1)\,4=229$; whence $n=56$.

14. $16a-8b+(n-1)\,(-a+b)=8a$; whence $n=9$.

16. 8 is the first term and 29 is the *eighth* term of an A. P. Hence d is given by $8+7d=29$. Thus $d=3$, and the means are 11, 14, &c.

17. 50 is the first and 80 is the tenth term of an A. P. Hence d is given by $50+9d=80$. Thus $d=3\frac{1}{3}$, and the means are $53\frac{1}{3}$, $56\frac{2}{3}$, &c.

18. $269+8d=295$; $\therefore d=3\frac{1}{4}$. Hence means are $272\frac{1}{4}$, $275\frac{1}{2}$, &c.

19. $67+16d=43$; $\therefore d=-1\frac{1}{2}$. Hence means are $65\frac{1}{2}$, 64, &c.

20. $84+26d=40\frac{2}{3}$; $\therefore d=-1\frac{2}{3}$. Hence the means are $82\frac{1}{3}$, $80\frac{2}{3}$, &c.

21. $5a-6b+11d=5b-6a$; $\therefore d=b-a$. Hence the means are $4a-5b$, $3a-4b$, &c.

23. By definition, $b-a=d-c$; $\therefore b+c=a+d$.

24. We have $a+(a+3d)=19$, and $(a+2d)+(a+5d)=31$, &c.

25. We have $(a+d)+(a+4d)=32$, and $(a+2d)+(a+7d)=48$, &c.

26. We have $(a+2d)+(a+3d)=187$, and $(a+6d)+(a+7d)=147$.
Hence $a=106$ and $d=-5$; $\therefore$ the second term $=106-5=101$.

27. We have $(a+d)+(a+19d)=2$, and $(a+8d)+(a+14d)=8$.
Hence $a=-29$, $d=3$; $\therefore (a+5d)+(a+6d)=-25$.

28. Let a, $a+d$, $a+2d$, $a+3d$, ... be any series of quantities in A. P.
Then $a+x$, $a+d+x$, $a+2d+x$, $a+3d+x$, ...
are also in A. P., the common difference being the same as before.

29. Multiplying by x each term of the series a, $a+d$, $a+2d$, $a+3d$, ..., we obtain the series

$$ax,\ ax+dx,\ ax+2dx,\ ax+3dx,\ \ldots,$$

which is obviously an A. P. whose common difference is dx.

30. Taking away the alternate terms of the series

$$a,\ a+d,\ a+2d,\ a+3d,\ \ldots$$

we obtain the series a, $a+2d$, $a+4d$, ...,

which is obviously an A. P. whose common difference is $2d$.

31. If between every two terms of the A. P.

$$a, \ a+d, \ a+2d, \ a+3d, \ldots$$

their arithmetic mean is inserted, we obtain the series

$$a, \ a+\frac{1}{2}d, \ a+d, \ a+\frac{3}{2}d, \ a+2d, \ldots,$$

which is an A. P. whose common difference is $\frac{1}{2}d$.

32. Let the four quantities be $a, \ a+d, \ a+2d, \ a+3d$.

Then $a(a+3d)<(a+d)(a+2d)$, if $a^2+3ad<a^2+3ad+2d^2$, which is obvious.

LXIV. Pages 271—273.

1. to **13.** Use the formula $S=\frac{n}{2}\{2a+(n-1)d\}$.

14. The series is $(\sqrt{2}-1)+\sqrt{2}+(\sqrt{2}+1)+\ldots$. Thus $a=\sqrt{2}-1$, $d=1$, $n=7$. Hence $S=\frac{7}{2}\{2(\sqrt{2}-1)+6\times 1\}=7\sqrt{2}+14$.

15. $a=\frac{n-1}{n}$, $d=-\frac{1}{n}$. Hence $S=\frac{n}{2}\left\{2\frac{n-1}{n}+(n-1)\left(-\frac{1}{n}\right)\right\}=\&c.$

16. $a=n+1$, $d=n+2$. Hence $S=\frac{n}{2}\{2(n+1)+(n-1)(n+2)\}=\&c.$

17. The first term is $(a+b)^2$, the common difference is $-2ab$. Hence

$$S=\frac{n}{2}\{2(a+b)^2+(n-1)(-2ab)\}=\&c.$$

18. $a+2d=15$ and $a+19d=23\frac{1}{2}$; whence $a=14$, $d=\frac{1}{2}$;

$$\therefore S=\frac{20}{2}\left\{2\times 14+19\times\frac{1}{2}\right\}=\&c.$$

19. $a+4d=37$ and $a+12d=81$; whence $a=15$, $d=\frac{11}{2}$. Hence

$$S=\frac{24}{2}\left\{2\times 15+23\times\frac{11}{2}\right\}=\&c.$$

20. The series is 25, 27, 29, &c. Hence $S=\frac{20}{2}\{2\times 25+19\times 2\}=\&c.$

21. The series is 99, 97, 95, &c. Hence

$$S=\frac{40}{2}\{2\times 99+39(-2)\}=\&c.$$

22. In the series whose 1st term is 5 and whose 31st term is 50,

$$5+30d=50\,;\ \therefore\ d=1\tfrac{1}{2}.$$

Hence the means are $6\frac{1}{2}$, 8, $9\frac{1}{2}$, &c. Hence

$$S=\frac{29}{2}\left\{2\times\frac{13}{2}+28\times\frac{3}{2}\right\}=\&c.$$

23. In the series whose 1st term is 10 and whose 42nd term is 100, $10+41d=100$; $\therefore\ d=\frac{90}{41}=2\,\frac{8}{41}$. Thus the means are $12\frac{8}{41}$, $14\frac{16}{41}$, &c. Hence $S=\frac{40}{2}\left\{2\times12\frac{8}{41}+39\times2\frac{8}{41}\right\}=\&c.$

24. $S=\frac{27}{2}(3+107)=1485.$

The middle term is the 14th$=a+13d$, where $a=3$, $a+26d=107$.

25. $S=\frac{71}{2}(7+1015)=36281.$

The middle term is the 36th$=a+35d$, where $a=7$ and $a+70d=1015$.

26. $S=\frac{n}{2}(a+l)=\frac{19}{2}(8-4)=38.$

27. The 7th term is $3+6\times2=15$. Hence

$$S=\frac{20}{2}\{2\times15+19\times2\}=\&c.$$

28. The 5th term is $6+4\times3=18$. Hence

$$S=\frac{35}{2}\{2\times18+34\times3\}=\&c.$$

29. $155=\frac{10}{2}\{2\times2+9d\}$, whence $d=3$.

30. $25=\frac{25}{2}\{2\times6+24d\}$, whence $d=-\frac{5}{12}$.

31. $100=\frac{10}{2}(2a+9d)$, and $a+5d=11$.

32. $133=\frac{28}{2}(2a+27d)$, and $a+4d=0$.

33. $S=\frac{n}{2}\{2a+(n-1)d\}$. But $S=705$, $n=10$, $d=5$;

$$\therefore\ 705=5(2a+45)\,;\ \therefore\ a=48.$$

34. $1025=\frac{25}{2}(2a+24\times3)$; $\therefore\ a=5$.

35. $a=\frac{4}{3}$, $d=-\frac{1}{3}$, $S=0$. Hence

$$0=\frac{n}{2}\left\{2\times\frac{4}{3}+(n-1)\left(-\frac{1}{3}\right)\right\}\ ;\ \therefore\ n-1=8.$$

36. $S=45$, $a=15$, $d=-3$. Hence $45=\frac{n}{2}\{2\times 15+(n-1)(-3)\}$;

$$\therefore\ n^2-11n+30,\ \text{whence}\ n=5\ \text{or}\ 6.$$

37. $-6a=\frac{n}{2}\left\{2\left(1-\frac{1}{a}\right)+(n-1)\left(-\frac{2}{a}\right)\right\}$,

whence $n^2-an-6a^2=0$, whence $n=3a$ or $n=-2a$, of which solutions neither has any meaning unless it is both positive and integral.

38. $42=\frac{n}{2}\{-8\times 2+(u-1)\,1\}$;

$$\therefore\ n^2-17n-84=0,\ i.e.\ (n-21)(n+4)=0.$$

Hence $n=21$, the solution $n=-4$ being meaningless.

39. $S=\frac{n}{2}(a+l)$; $\therefore\ 450=\frac{n}{2}(1+99)$; $\therefore\ n=9$.

40. $670=\frac{20}{2}(a+62)$; $\therefore\ a=5$.

Then $$62=a+19d=5+19d\ ;\ \therefore\ d=3.$$

41. $a+8d=136$. Also $2527=\frac{19}{2}\{2a+18d\}$; $\therefore\ a+9d=133$.

Hence $$d=-3,\ a=160.$$

Hence $$S=\frac{40}{2}\{2\times 160+39\,(-3)\}=\&c.$$

42. $S=\frac{15}{2}\{2a+14d\}$; and $a+7d=6$. Hence $S=15\times 6=90$.

43. $S=\frac{35}{2}\{2a+34d\}$; and $a+17d=15$. Hence $S=35\times 15=525$.

44. The $(n+1)$th term is $a+nd=100$. The sum of $(2n+1)$ terms is

$$\frac{2n+1}{2}\{2a+2nd\}=(2n+1)\,(a+nd)=(2n+1)\times 100.$$

45. Let there be $2n+1$ terms of which the first is a, and the common difference d. Then the first, the middle and the last terms are respectively a, $a+nd$ and $a+2nd$, which are in A.P.

46. The sum of n terms $=\frac{n}{2}\{2\times 8+(n-1)\,8\}=4n^2+4n$.

And $$(4n^2+4n)+1=(2n+1)^2.$$

47. The numbers are 101, 103, ..., 199. Since $199=101+(n-1)\,2$, $n=50$.

Then $$S=\frac{50}{2}(101+199)=7500.$$

48. The numbers are 102, 104, ..., 998. Since
$$998=102+(n-1)\,2,\quad n=449.$$

Then $$S=\frac{449}{2}(102+998)=246950.$$

49. The numbers are 102; 105, ..., 498.

Since $$498=102+(n-1)\,3,\quad n=133.$$

Then $$S=\frac{133}{2}(102+498)=39900.$$

50. $1700=20+30+\ldots$ to n terms $=\frac{n}{2}\{40+(n-1)\,10\}$. Hence
$$n^2+3n-340=0,\ i.e.\ (n-17)(n+20)=0.$$
Thus $n=17$, the negative value being meaningless.

51. $145=\frac{10}{2}(2a+9d)$; also $(a+3d)+(a+8d)=5\,(a+2d)$.

Hence $2a+9d=29$ and $3a=d$; whence $a=1$, $d=3$.

52. Let the series be a, $a+d$, $a+2d$, ..., $a+2nd$.
The sum of the $n+1$ odd terms, namely a, $a+2d$, ..., is
$$\frac{n+1}{2}\{2a+n\times 2d\}=(n+1)(a+nd).$$

The sum of the n even terms, namely $a+d$, $a+3d$, ..., is
$$\frac{n}{2}\{2\,(a+d)+(n-1)\,2d\}=n\,(a+nd).$$

53. $$(a+20d)+(a+19d)+(a+18d)=117;$$
and $$(a+11d)+(a+10d)+(a+9d)=90.$$
Whence $$d=1 \text{ and } a=20.$$

54. $\frac{n}{2}\{2\times 87+(n-1)(-2)\}=\frac{n}{2}\{2\times 3+(n-1)\,2\}$;
$$\therefore\ 176-2n=4+2n;\ \therefore\ n=43.$$

55. $\frac{n}{2}\{43\times 2+(n-1)\,2\}=\frac{2n}{2}\{45\times 2+(2n-1)(-2)\}$;
$$\therefore\ 84+2n=184-8n;\ \therefore\ n=10.$$

56. Let the parts be $a+3d$, $a+d$, $a-d$ and $a-3d$.
Then $$(a+3d)+(a+d)+(a-d)+(a-3d)=80,$$
and $$(a+3d)(a-3d)=\frac{2}{3}(a+d)(a-d).$$

Hence $a=20$ and $3(a^2-9d^2)=2(a^2-d^2)$;

$$\therefore\ 25d^2=a^2=400;\ \therefore\ d=\pm4.$$

Thus the parts are 32, 24, 16 and 8.

57. Let the numbers be $a+3d$, $a+d$, $a-d$, $a-3d$. Then

$$(a+3d)^2+(a+d)^2+(a-d)^2+(a-3d)^2=120,$$

and

$$(a+3d)(a-3d)=(a+d)(a-d)-8.$$

Hence $a^2+5d^2=30$; and $d^2=1$. Whence $a^2=25$.

Thus the numbers are $\pm5\pm3$, $\pm5\pm1$, $\pm5\mp1$, $\pm5\mp3$; that is ±8, ±6, ±4, ±2, or the same numbers in reverse order.

58. $(s-a)^2$, $(s-b)^2$, $(s-c)^2$ are in A. P. if

$$(s-b)^2-(s-a)^2=(s-c)^2-(s-b)^2;$$

or if $\{(s-b)+(s-a)\}\{(s-b)-(s-a)\}=\{(s-c)+(s-b)\}\{(s-c)-(s-b)\}$;

or if

$$c(a-b)=a(b-c);\ \text{or, if}\ \frac{a-b}{ab}=\frac{b-c}{bc};$$

or, if $\frac{1}{b}-\frac{1}{a}=\frac{1}{c}-\frac{1}{b}$; that is, if $\frac{1}{a}, \frac{1}{b}, \frac{1}{c}$ are in A. P.

59. $a^2(b+c)$, $b^2(c+a)$, $c^2(a+b)$ are in A. P. if

$$c^2(a+b)-b^2(c+a)=b^2(c+a)-a^2(b+c);$$

or, if

$$a(c^2-b^2)+bc(c-b)=c(b^2-a^2)+ab(b-a);$$

or, if

$$(c-b)(bc+ca+ab)=(b-a)(bc+ca+ab);$$

or, if $c-b=b-a$, that is, if a, b, c are in A. P.

LXV. Pages 277, 278.

1. (i) $9\times\left(\frac{1}{3}\right)^5$. (ii) $2\times\left(-\frac{3}{2}\right)^5$. (iii) $a^2\times\left(\frac{b}{a}\right)^5$.

2. $a=3$, $ar^2=4$. Hence $ar^4=\frac{16}{3}$.

3. $ar^2=1$, $ar^5=-\frac{1}{8}$. Hence $ar^9=-\frac{1}{128}$.

4. $ar^3=\cdot016$, $ar^6=\cdot000128$; $\therefore\ a=(\cdot016)^2\div\cdot000128=2$.

5. $ar^5=156$, $ar^7=7644$; $ar^6=\sqrt{156\times7644}=\pm1092$.

6. $ar^2=2\cdot25$ and $ar^6=11\cdot390625$;

$$\therefore\ ar^4=\sqrt{(2\cdot25\times11\cdot390625)}=\pm5\cdot0625.$$

7. $ar^2=4$ and $ar^5=-\frac{1}{2}$; $\therefore\ r=-\frac{1}{2}$. Then $ar^9=-\frac{1}{32}$.

8. $ar^3=\frac{1}{18}$ and $ar^6=-\frac{1}{486}$; $\therefore\ r=-\frac{1}{3}$. Then $ar^5=\frac{1}{162}$.

9. $\sqrt{4\times 9}$, $\sqrt{7\times 252}$, $\sqrt{a^3b\times ab^3}$.

10. (i) $a=1$, $ar^3=-8$; $\therefore r=-2$. Hence the means are -2, 4.

(ii) $a=12$, $ar^4=\frac{3}{4}$; $\therefore r^4=\frac{1}{16}$, or $r=\pm\frac{1}{2}$. Hence the means are ± 6, 3, $\pm\frac{3}{2}$.

(iii) $a=\frac{2}{3}$, $ar^5=\frac{81}{16}$; $\therefore r^5=\frac{243}{32}$; $\therefore r=\frac{3}{2}$. Hence the means are 1, $\frac{3}{2}$, $\frac{9}{4}$, $\frac{27}{8}$.

11. Here $a=3$, $ar^6=\cdot 000192$; $\therefore r=\pm\cdot 2$. Thus the means are $\pm\cdot 6$, $\cdot 12$, $\pm\cdot 024$, $\cdot 0048$, $\pm\cdot 00096$.

12. $a^{-2}b^4=a^3b^{-6}\times r^5$; $\therefore r=a^{-1}b^2$. Hence the means are a^2b^{-4}, ab^{-2}, 1, $a^{-1}b^2$.

13. $ar^2+ar^3=40$ and $ar^5+ar^6=2560$. By division, $r^3=64$; $\therefore r=4$. Hence $a=40\div(16+64)=\frac{1}{2}$.

14. $a+ar=72$ and $ar^2+ar^3=8$; $\therefore r^2=\frac{1}{9}$; $r=\pm\frac{1}{3}$. Hence

$$a=72\div\left(1\pm\frac{1}{3}\right)=54 \text{ or } 108.$$

15. By definition $\frac{b}{a}=\frac{d}{c}$; $\therefore bc=ad$.

16. Let the original series be a, ar, ar^2,.... Then the series ax, axr, axr^2,... is also a G. P. with the same common ratio.

17. Let the G.P. be a, ar, ar^2,.... Then the reciprocals of the terms form the series $\frac{1}{a}$, $\frac{1}{ar}$, $\frac{1}{ar^2}$,... which is a G.P. whose common ratio is $\frac{1}{r}$.

18. Let the original series be a, ar, ar^2, ar^3,.... Then after taking away the alternate terms we have a, ar^2, ar^4,..., which is a G.P. whose common ratio is r^2.

19. Let the original G.P. be a, ar, ar^2, ar^3,.... Then, after inserting the geometric means, we have the series

$$a,\ ar^{\frac{1}{2}},\ ar,\ ar^{\frac{3}{2}},\ ar^2,\ldots,$$

which is a G.P. whose common ratio is $r^{\frac{1}{2}}$.

20. Let the series be a, ar, ar^2,..., ar^n. Then the pth term from the beginning is ar^{p-1}, and the pth term from the end is $ar^{n-(p-1)}$; the product is therefore $ar^{p-1}\times ar^{n-(p-1)}=a^2r^n=a\times ar^n$.

LXVI. Pages 282, 283.

1. $8\,\dfrac{\left(\frac{4}{3}\right)^n-1}{\frac{4}{3}-1}=24\left\{\left(\frac{4}{3}\right)^n-1\right\}.$

2. $12\,\dfrac{1-\left(\frac{3}{4}\right)^6}{1-\frac{3}{4}}=48\left\{1-\left(\frac{3}{4}\right)^6\right\}.$

3. $a\,\dfrac{1-\left(\frac{1}{r}\right)^n}{1-\frac{1}{r}}=\dfrac{a\,(r^n-1)}{r^{n-1}\,(r-1)}.$

4. $\dfrac{3}{2}\cdot\dfrac{1-\left(\frac{8}{9}\right)^8}{1-\frac{8}{9}}=\dfrac{27}{2}\left\{1-\left(\frac{8}{9}\right)^8\right\}.$

5. $\dfrac{2}{9}\,\dfrac{\left(\frac{3}{2}\right)^6-1}{\frac{3}{2}-1}=\dfrac{4}{9}\left\{\left(\frac{3}{2}\right)^6-1\right\}.$

6. $\dfrac{12}{1-\left(-\frac{3}{4}\right)}=\dfrac{48}{7}.$

7. $\dfrac{4}{1-\frac{3}{10}}=\dfrac{40}{7}.$

8. $\dfrac{4}{1-\frac{1}{5}}=5.$

9. $a^2\,\dfrac{1-\left(\frac{b}{a}\right)^n}{1-\frac{b}{a}}=\dfrac{a^n-b^n}{a^{n-3}\,(a-b)}.$

10. $\dfrac{3}{1-\left(-\frac{2}{3}\right)}=\dfrac{9}{5}.$

11. $S=\dfrac{a-rl}{1-r}=\dfrac{2-4096\times 2}{1-2}=8190.$

12. $S=\dfrac{1}{1-\frac{1}{\sqrt{2}}}=\dfrac{\sqrt{2}}{\sqrt{2}-1}=\sqrt{2}\,(\sqrt{2}+1)=2+\sqrt{2}=2+1{\cdot}4142\ldots$

13. The geometric mean $=\sqrt{\{(4x^2-12x+9)\times(9x^2+12x+4)\}}$

$$=\pm(2x-3)\,(3x+2).$$

14. Let the $2m+1$ terms be a, ar, ar^2, $\ldots$, ar^{2m}.

The continued product $=a\times ar\times ar^2\times\ldots\times ar^{2m}=a^{2m+1}r^{1+2+\ldots+2m}$

$$=a^{2m+1}r^{m(2m+1)}=(ar^m)^{2m+1}.$$

But ar^m is the middle term of the $2m+1$ terms, and $2m+1=n$. Hence the product is equal to the nth power of the middle term.

15. $4=\dfrac{a}{1-r}$ and $ar=\dfrac{3}{4}$. Hence $4\,(1-r)\,r=ar=\dfrac{3}{4}$;

$$\therefore\ 16r^2-16r+3=0,\ \textit{i.e.}\ (4r-3)\,(4r-1)=0.$$

Hence $r=\frac{1}{4}$ or $\frac{3}{4}$; and then $a=3$ or 1.

Thus the series is $3, \frac{3}{4}, \frac{3}{16}, \dots$ or $1, \frac{3}{4}, \frac{9}{16}, \dots$

16. $9=\frac{a}{1-r}$ and $ar=-4$. Hence $9(1-r)r=ar=-4$;

$$\therefore 9r^2-9r-4=0; \quad \therefore r=-\frac{1}{3} \text{ or } r=\frac{4}{3}.$$

[The latter value is not, however, applicable to the question, since it is greater than 1.]

From $r=-\frac{1}{3}$, we find $a=12$. Hence the series is $12, -4, \frac{4}{3}$, &c.

17. $a\frac{1-r^{10}}{1-r}=a\frac{1-r^5}{1-r}\times 33$; $\therefore 1+r^5=33$; whence $r=2$.

18. $\frac{a}{1-r}=na$; $\therefore 1-r=\frac{1}{n}$, or $r=1-\frac{1}{n}$.

19. Let a be any term of a G. P., and let r be the common ratio. Then $a>ar+ar^2+\dots$ to infinity, if $a>\frac{ar}{1-r}$, or if $1-r>r$, or $r<\frac{1}{2}$.

20. $ar^5=8ar^2$, and $a+ar=24$; $\therefore r^3=8$, and then $a=8$.

21. $\dfrac{\frac{a}{a+b}}{1-\frac{a}{a+b}}=\frac{a}{b}$.

22. $\dfrac{1}{1-\left(1-\frac{1}{a}\right)}=a$.

23. $S_1=a\frac{1-r^{2n+1}}{1-r}$, $S_2=a\frac{1-(-r)^{2n+1}}{1-(-r)}=a\frac{1+r^{2n+1}}{1+r}$. Hence

$$S_1S_2=a^2\frac{1-r^{4n+2}}{1-r^2}=a^2+a^2r^2+\dots+a^2r^{4n}.$$

24. $\dfrac{\sqrt{\frac{3}{2}}}{1-\frac{2}{3\sqrt{3}}}=\frac{9}{\sqrt{2}(3\sqrt{3}-2)}=\frac{9}{46}(3\sqrt{6}+2\sqrt{2})$.

25. $ar^5=\frac{1}{33614}$ and $ar=\frac{1}{14}$.

$$\therefore r=\pm\frac{1}{7}. \quad \text{Hence } \frac{a}{1-r}=\frac{\pm\frac{1}{2}}{1\mp\frac{1}{7}}=\frac{7}{12} \text{ or } -\frac{7}{16}.$$

26. $a+ar+ar^2=14$, $a^2+a^2r^2+a^2r^4=84$. By division $a-ar+ar^2=6$. Hence $a=2$ or 8, $r=2$ or $\frac{1}{2}$. Thus the numbers are 2, 4, 8.

27. $a+ar+ar^2=42$ and $ar^3-a=126$. By division $r-1=3$; $\therefore\ r=4$ and then $a=2$.

28. $a+ar+ar^2 : ar^2+ar^3+ar^5=1 : 4$; $\therefore\ r=\pm 2$.

Also
$$ar^4=8;\ \therefore\ a=\frac{1}{2}.$$

29. If a, b, c, d are in G. P., $b=ar$, $c=ar^2$, $d=ar^3$. Hence $a+b$, $b+c$, $c+d$ will be $a(1+r)$, $ar(1+r)$, $ar^2(1+r)$, which are clearly in G. P.

Also a^2+b^2, b^2+c^2, c^2+d^2 will be $a^2(1+r^2)$, $a^2r^2(1+r^2)$, $a^2r^4(1+r^2)$, which are clearly in G. P.

30. Let the G. P. be a, am, am^2, &c. Then $a^{q-r}b^{r-p}c^{p-q}$
$$=(am^{p-1})^{q-r}\times(am^{q-1})^{r-p}\times(am^{r-1})^{p-q}$$
$$=a^{q-r+r-p+p-q}\times m^{(p-1)(q-r)+(q-1)(r-p)+(r-1)(p-q)}$$
$$=a^0m^0=1.$$

31. If $(a^2+b^2)(b^2+c^2)=(ab+bc)^2$, then $a^2c^2-2ab^2c+b^4=0$;
$$\therefore\ ac-b^2=0.\quad \text{Hence } \frac{b}{a}=\frac{c}{b}.$$

LXVII. Pages 284, 285.

1. This is an A. P. since $2{\cdot}7-3=2{\cdot}4-2{\cdot}7$.
$$S=\frac{21}{2}\left\{2\times 3+20\left(-\frac{3}{10}\right)\right\}=\&c.$$

2. This is a G. P. since $\frac{18}{2}=\frac{162}{18}$. $S=2\,\frac{9^7-1}{9-1}=\frac{1}{4}(9^7-1)$.

3. This is a G. P. $S=\frac{1}{1-{\cdot}2}=\frac{10}{8}=1\frac{1}{4}$.

4. This is a G. P. $S=a\,\dfrac{\left(\frac{b}{a}\right)^n-1}{\frac{b}{a}-1}=\&c.$

5. $S=\dfrac{{\cdot}3}{1-\frac{1}{10}}=\frac{1}{3}$.

6. $S=\frac{11}{2}\{2\times 3+10(1{\cdot}3)\}=\&c.$

7. $S=\frac{19}{2}\left\{2a+18\times\frac{a+b}{3}\right\}=\&c.$

8. $$S=\frac{1}{4}\,\frac{\left(-\frac{4}{3}\right)^8-1}{\left(-\frac{4}{3}\right)-1}=-\frac{3}{28}\left\{\left(\frac{4}{3}\right)^8-1\right\}.$$

9. $$S=\frac{3\frac{1}{8}}{1-\frac{2}{5}}=\frac{125}{24}.$$

10. $$S=2\tfrac{1}{2}\,\frac{\left(\frac{5}{2}\right)^5-1}{\frac{5}{2}-1}=\&c.$$

11. (i) $$S=\frac{6}{2}\left\{2\times 2\tfrac{1}{2}+5\times\frac{15}{4}\right\}=\&c.$$ (ii) $$S=\frac{6}{2}\{2\times 9+5\,(-4)\}.$$

(iii) $$S=4\,\frac{1-\left(-\frac{3}{4}\right)^6}{1-\left(-\frac{3}{4}\right)}=\frac{16}{7}\left\{1-\left(\frac{3}{4}\right)^6\right\}.$$ Sum to infinity $\frac{16}{7}$.

(iv) $$S=\frac{1}{3}\,\frac{\left(\frac{3}{2}\right)^6-1}{\frac{3}{2}-1}=\frac{2}{3}\left\{\left(\frac{3}{2}\right)^6-1\right\}.$$

(v) $$S=\frac{1}{2}\,\frac{1-\left(\frac{2}{3}\right)^6}{1-\frac{2}{3}}=\frac{3}{2}\left\{1-\left(\frac{2}{3}\right)^6\right\}.$$ Sum to infinity $\frac{3}{2}$.

12. Let the series be a, ar, ar^2, ... Then, if there are $2n+1$ terms, the middle term will be the $(n+1)$th term. Hence the first, the middle and the last terms will be a, ar^n, ar^{2n}, which are obviously in G. P.

13. We have $\frac{7}{2}\{2a+6d\}=49$; also $\frac{15}{2}\{2a+14d\}=49+176$, &c.

14. Let the series be a, ar, ar^2, &c. Then $\frac{1}{2}(a+ar^2)=5ar$, whence

$$r^2-10r+1=0;\ \therefore\ r=5\pm\sqrt{24}.$$

15. Let the series be a, $a+d$, $a+2d$, &c. Then

$$a+3d=\sqrt{\{(a+d)\,(a+6d)\}};$$

$\therefore\ a=3d$ [or $d=0$, which make all the terms alike]. Since

$$a=3d,\ (a+5d)^2=(a+d)\,(a+13d).$$

16. Let r be the common ratio; then $P=a\,.\,ar\,.\,ar^2\,.\,ar^3\ldots\ldots ar^{n-1}$;

also $$P=l\,.\,\frac{l}{r}\,.\,\frac{l}{r^2}\,.\,\frac{l}{r^3}\,\ldots\ldots\,\frac{l}{r^{n-1}}.$$ Hence $P^2=(al)^n$.

17. We have $a \cdot ar \cdot ar^2 = 216$, and $a \cdot ar + a \cdot ar^2 + ar \cdot ar^2 = 156$.

From the first $a^3r^3 = 216$; $\therefore ar = 6$.

Hence $\frac{36}{r} + 36 + 36r = 156$, whence $r = 3$ or $\frac{1}{3}$.

18. Let the parts be $a - 2d$, $a - d$, a, $a + d$, $a + 2d$;

then $(a-2d) + (a-d) + a + (a+d) + (a+2d) = 25$; $\therefore a = 5$.

Also $(a-2d)^2 + (a+2d)^2 = (a-d)^2 + a^2 + (a+d)^2 - 1$;

$\therefore 6d^2 = a^2 - 1 = 25 - 1$; $\therefore d = \pm 2$.

Thus the numbers are 9, 7, 5, 3, 1.

19. If 6, x, y are in A. P., $2x = 6 + y$. If x, y, 16 are in G. P., $y^2 = 16x$. Hence $y^2 = 48 + 8y$, whence $y = 12$ or -4. Then $x = 9$ or 1. Thus the numbers are 6, 9, 12, 16, or 6, 1, -4, 16.

20. If a, b, c be in G.P.; then $b^2 = ac$. Also $2x = a + b$ and $2y = b + c$.

Hence $b^2 = ac = (2x - b)(2y - b)$; $\therefore 0 = 4xy - 2by - 2bx$; $\therefore \frac{2}{b} = \frac{1}{x} + \frac{1}{y}$.

Again $ac = b^2 = (2x - a)(2y - c)$; $\therefore 4xy - 2ay - 2cx = 0$; $\therefore 2 = \frac{a}{x} + \frac{c}{y}$.

LXVIII. PAGES 289—291.

1. For, if $a - b : b - c :: a : c$, then $ma - mb : mb - mc :: ma : mc$.

2. Insert 5 arithmetic means between 1 and $\frac{1}{7}$, and then invert.

3. Insert 4 arithmetic means between $\frac{3}{2}$ and $\frac{2}{3}$, and then invert.

4. For $AH = G^2$ [Art. 228]. Hence, if $A = 1$, $H = G^2$.

5. $b + c$, $c + a$, $a + b$ are in H.P.,

if $(b+c) - (c+a) : (c+a) - (a+b) = b + c : a + b$,

i.e., if $b - a : c - b = b + c : a + b$, *i.e.*, if $(b-a)(a+b) = (c-b)(b+c)$,

i.e., if $b^2 - a^2 = c^2 - b^2$, *i.e.*, if a^2, b^2, c^2 are in A. P.

6. $(y + z - x)^2$, $(z + x - y)^2$, $(x + y - z)^2$ are in A.P.,

if $(x+y-z)^2 - (z+x-y)^2 = (z+x-y)^2 - (y+z-x)^2$,

i.e., if $2x(2y - 2z) = 2z(2x - 2y)$, *i.e.*, if $x(y - z) = z(x - y)$,

i.e., if $x - y : y - z = x : z$, *i.e.*, if x, y, z, are in H. P.

7. $\frac{1}{x}, \frac{1}{y}, \frac{1}{z}$, are in A.P.; $\therefore$ $\frac{x+y+z}{x}, \frac{x+y+z}{y}, \frac{x+y+z}{z}$ are in A.P.;

$$\therefore \frac{x+y+z}{x}-1, \frac{x+y+z}{y}-1, \frac{x+y+z}{z}-1 \text{ are in A.P.,}$$

$$i.e., \frac{y+z}{x}, \frac{z+x}{y}, \frac{x+y}{z} \text{ are in A.P.; } \therefore \text{ \&c.}$$

8. As in 7, $\frac{x+y+z}{x}, \frac{x+y+z}{y}, \frac{x+y+z}{z}$ are in A.P.;

$$\therefore \frac{x+y+z}{x}-2, \frac{x+y+z}{y}-2, \frac{x+y+z}{z}-2 \text{ are in A.P.,}$$

$$i.e. \frac{y+z-x}{x}, \frac{z+x-y}{y}, \frac{x+y-z}{z} \text{ are in A.P., } \therefore \text{ \&c.}$$

9. As in 7, $\frac{a+b+c+d}{a}, \frac{a+b+c+d}{b}, \frac{a+b+c+d}{c}, \frac{a+b+c+d}{d}$ are in A.P.;

$$\therefore \frac{b+c+d}{a}, \frac{c+d+a}{b}, \frac{d+a+b}{c}, \frac{a+b+c}{d} \text{ are in A.P.; } \therefore \text{ \&c.}$$

10. $2a-b$, b, $2c-b$ are in G.P., if $(2a-b)(2c-b)=b^2$,

or if $4ac=2bc+2ab$, or if $\frac{2}{b}=\frac{1}{a}+\frac{1}{c}$; that is, if a, b, c are in H.P.

11. a, $a-c$, $a-b$ are in H.P.,

if $a-(a-c) : (a-c)-(a-b)=a : a-b$, [Def.

i.e., if $c : b-c=a : a-b$, *i.e.*, if $a-b : b-c=a : c$,

i.e., if a, b, c are in H.P.

12. a^2-ab+b^2, a^2+b^2, a^2+ab+b^2 are obviously in A.P.,

$$\therefore \frac{1}{a^2-ab+b^2}, \frac{1}{a^2+b^2}, \frac{1}{a^2+ab+b^2} \text{ are in H.P.}$$

Multiply each of these terms by $(a^2-ab+b^2)(a^2+b^2)(a^2+ab+b^2)$; then the products will be in H.P., and these products are

$$(a^2+b^2)(a^2+ab+b^2),\ a^4+a^2b^2+b^4,\ (a^2+b^2)(a^2-ab+b^2).$$

13. By Def. $a_1-x=y-a_2$; $\therefore$ $a_1+a_2=x+y$.

By Def. $\frac{g_1}{x}=\frac{y}{g_2}$; $\therefore$ $g_1g_2=xy$.

Also $\frac{1}{x}, \frac{1}{h_1}, \frac{1}{h_2}, \frac{1}{y}$ are in A.P., thus

by Def. $\frac{1}{h_1}-\frac{1}{x}=\frac{1}{y}-\frac{1}{h_2}$; $\therefore$ $\frac{1}{h_1}+\frac{1}{h_2}=\frac{1}{x}+\frac{1}{y}$;

$$i.e. \frac{h_1+h_2}{h_1h_2}=\frac{x+y}{xy}=\frac{a_1+a_2}{g_1g_2}; \therefore \frac{g_1g_2}{h_1h_2}=\frac{a_1+a_2}{h_1+h_2}.$$

14. $\dfrac{b+a}{b-a}+\dfrac{b+c}{b-c}=2$, if $\dfrac{b+c}{b-c}-1=1+\dfrac{a+b}{a-b}$,

i.e., if $\dfrac{2c}{b-c}=\dfrac{2a}{a-b}$, *i.e.* if $(a-b)\,c=(b-c)\,a$,

i.e. if $a-b : b-c=a : c$, *i.e.* if a, b, c are in H.P.

15. These are in A.P., if $\dfrac{1}{b}-\dfrac{2}{a+b}=\dfrac{2}{b+c}-\dfrac{1}{b}$,

i.e. if $\dfrac{a-b}{b\,(a+b)}=\dfrac{b-c}{b\,(b+c)}$, *i.e.* if $(a-b)\,(b+c)=(a+b)\,(b-c)$,

i.e. if $ac=b^2$, *i.e.* if a, b, c are in G.P.

16. a, b, c are in A.P.; $\therefore\ b-a=c-b$; whence $a=2b-c$.

$\dfrac{1}{b}, \dfrac{1}{c}, \dfrac{1}{d}$ are in A.P.; $\therefore\ \dfrac{1}{c}-\dfrac{1}{b}=\dfrac{1}{d}-\dfrac{1}{c}$; whence $d=\dfrac{bc}{2b-c}$.

$$\therefore\ ad=(2b-c)\left(\frac{bc}{2b-c}\right)=bc;\quad \therefore\ a : b=c : d.$$

17. $\dfrac{1}{a}-\dfrac{1}{b-c}=\dfrac{1}{b-a}-\dfrac{1}{c}$; whence $\dfrac{b-(c+a)}{a\,(b-c)}=\dfrac{b-(c+a)}{c\,(a-b)}$;

thus, either $b-(c+a)=0$; *i.e.* $b=c+a$;

or, $\dfrac{1}{a\,(b-c)}=\dfrac{1}{c\,(a-b)}$; $\therefore\ a\,(b-c)=c\,(a-b)$;

$$\therefore\ a-b : b-c=a : c;\quad \therefore\ a,\ b,\ c \text{ are in H. P.}$$

18. $\dfrac{1}{a}, \dfrac{1}{b}, \dfrac{1}{c}, \dfrac{1}{d}$ are in A.P.; $\therefore\ \dfrac{1}{d}-\dfrac{1}{c}=\dfrac{1}{c}-\dfrac{1}{b}=\dfrac{1}{b}-\dfrac{1}{a}$;

$$\therefore\ \frac{1}{d}-\frac{1}{a}=\left(\frac{1}{d}-\frac{1}{c}\right)+\left(\frac{1}{c}-\frac{1}{b}\right)+\left(\frac{1}{b}-\frac{1}{a}\right)=3\left(\frac{1}{b}-\frac{1}{a}\right);$$

$$i.e.\ \frac{a-d}{ad}=3\,\frac{a-b}{ab}.$$

Also $\dfrac{1}{c}-\dfrac{1}{b}=\dfrac{1}{d}-\dfrac{1}{c}$; *i.e.* $\dfrac{b-c}{bc}=\dfrac{c-d}{cd}$.

Hence $\dfrac{a-d}{ad}\cdot\dfrac{b-c}{bc}=3\,\dfrac{a-b}{ab}\cdot\dfrac{c-d}{cd}$;

$$\therefore\ (a-d)\,(b-c)=3\,(a-b)\,(c-d).$$

19. $\dfrac{2}{b}=\dfrac{1}{b-a}+\dfrac{1}{b-c}$, if $\dfrac{1}{b-a}-\dfrac{1}{b}+\dfrac{1}{b-c}-\dfrac{1}{b}=0$,

i.e. if $\dfrac{a}{b\,(b-a)}+\dfrac{c}{b\,(b-c)}=0$, *i.e.* if $a\,(b-c)=c\,(a-b)$,

i.e. if $a-b : b-c=a : c$, *i.e.* if a, b, c are in H. P.

20. Let the numbers be a, $a+d$, $a+2d$.
Then $a(a+2d)=5(a+d)$; and $(a+d)+(a+2d)=3a$.

From the second equation $a=3d$; and then $15d^2=20d$; $\therefore\ d=\frac{4}{3}$. Thus the numbers are 4, $5\frac{1}{3}$, $6\frac{2}{3}$.

21. We have $b-\frac{a+b}{1-ab}=\frac{b+c}{1-bc}-b$; $\therefore\ -\frac{a(b^2+1)}{1-ab}=\frac{c(b^2+1)}{1-bc}$;

$$\therefore\ \frac{a}{ab-1}=\frac{c}{1-bc};\ \therefore\ 2abc=a+c;\ \therefore\ 2b=\frac{1}{a}+\frac{1}{c},$$

which shews that a, $\frac{1}{b}$ and c are in H.P.

22. Since a, b, c are in A.P., and a^2, b^2, c^2 in H.P. we have,

$$2b=a+c \text{ and } \frac{2}{b^2}=\frac{1}{a^2}+\frac{1}{c^2};$$

$$\therefore\ \frac{8}{(a+c)^2}=\frac{1}{a^2}+\frac{1}{c^2};\ \therefore\ 8a^2c^2=(a+c)^2(a^2+c^2)=(a+c)^2\{(a+c)^2-2ac\};$$

$$\therefore\ (a+c)^4-2ac(a+c)^2-8a^2c^2=0.$$

Hence $(a+c)^2-4ac=0$ or $(a+c)^2+2ac=0$.
If $(a+c)^2-4ac=0$, $(a-c)^2=0$; $\therefore\ a=c$, and then each $=b$.
If $(a+c)^2+2ac=0$, $(2b)^2+2ac=0$; $\therefore\ -\frac{a}{2}$, b, c are in G.P.

23. Let the numbers be a, $a+d$, $a+2d$.
Then a, $a+d$, $a+2d+1$ are in G.P.

$$\therefore\ a(a+2d+1)=(a+d)^2, \text{ whence } a=d^2.$$

24. By Art. 215 and Art. 229

$$a=\frac{2x+y}{3},\ b=\frac{x+2y}{3};\ \text{also } u=\frac{3xy}{2y+x},\ v=\frac{3xy}{y+2x}.$$

$$\therefore\ av=\frac{2x+y}{3}\cdot\frac{3xy}{y+2x}=xy;\ \text{similarly } bu=xy.$$

25. $\frac{1}{a_1}$, $\frac{1}{a_2}$, $\frac{1}{a_3}$, $\frac{1}{a_4}$ are in A.P.;

$$\therefore\ \frac{1}{a_2}-\frac{1}{a_1}=\frac{1}{a_3}-\frac{1}{a_2}=\frac{1}{a_4}-\frac{1}{a_3}=(\text{say})\ x.$$

Thus $\frac{1}{a_4}-\frac{1}{a_1}=\left(\frac{1}{a_4}-\frac{1}{a_3}\right)+\left(\frac{1}{a_3}-\frac{1}{a_2}\right)+\left(\frac{1}{a_2}-\frac{1}{a_1}\right)=3x.$

Then $xa_1a_2=a_1-a_2$; $xa_2a_3=a_2-a_3$; $xa_3a_4=a_3-a_4$. Also $3xa_1a_4=a_1-a_4$.

$$\therefore\ x(a_1a_2+a_2a_3+a_3a_4)=3xa_1a_4.$$

26. a, b, c are in A.P., $\therefore \frac{a}{ab}, \frac{b}{ab}, \frac{c}{ab}$ are in A.P.

But $ab=c^2$, since a, c, b are in G.P.

$$\therefore \frac{a}{ab}, \frac{b}{ab}, \frac{c}{c^2} \text{ are in A.P., } i.e.\ \frac{1}{b}, \frac{1}{a}, \frac{1}{c} \text{ are in A.P.}$$

$$\therefore b, a, c \text{ are in H.P.}$$

LXIX. Page 295.

1. Let $\Sigma=2.4.6+4.6.8+6.8.10+\&c.\ldots+2n(2n+2)(2n+4)$,

thus $\Sigma=\quad 2.4.6+4.6.8+\&c.\ldots\ +(2n-2)2n(2n+2)$
$\qquad +2n(2n+2)(2n+4)$.

$\therefore$ subtracting,

$0=2.4.6+6.4.6+6.6.8+\&c.\ldots+6.2n(2n+2)-2n(2n+2)(2n+4)$.

$i.e.\ 0=6\{2.4+4.6+\&c.+2n(2n+2)\}-2n(2n+2)(2n+4)$.

$$\therefore S_n=\frac{4}{3}n(n+1)(n+2).$$

2. Let $\Sigma=3.5.7+5.7.9+7.9.11+\&c.\ldots+(2n+1)(2n+3)(2n+5)$,

then $\Sigma=\quad 3.5.7+5.7.9+\&c.\ldots\ +(2n-1)(2n+1)(2n+3)$
$\qquad +(2n+1)(2n+3)(2n+5)$.

$\therefore 0=3.5.7+6.5.7+6.7.9+\&c.\ldots\ +6(2n+1)(2n+3)$
$\qquad -(2n+1)(2n+3)(2n+5)$,

$i.e.\ 0=6\{3.5+5.7+\&c.\ldots+(2n+1)(2n+3)\}+3.5-(2n+1)(2n+3)(2n+5)$,

$$\therefore S_n=\frac{1}{6}\{(2n+1)(2n+3)(2n+5)-15\}.$$

3. Let $\Sigma=1.4.7+4.7.10+7.10.13+\&c.\ldots+(3n-2)(3n+1)(3n+4)$,

thus $\Sigma=\quad 1.4.7\ +4.7.10+\&c.\ldots\ +(3n-5)(3n-2)(3n+1)$
$\qquad +(3n-2)(3n+1)(3n+4)$.

$\therefore 0=1.4.7+9.4.7+9.7.10+\&c.\ldots\ +9(3n-2)(3n+1)$
$\qquad -(3n-2)(3n+1)(3n+4)$.

$i.e.\ 0=9\{1.4+4.7+\&c.\ldots+(3n-2)(3n+1)\}-8-(3n-2)(3n+1)(3n+4)$;

$$\therefore S_n=\frac{1}{9}\{(3n-2)(3n+1)(3n+4)+8\}.$$

4. $\Sigma=2.5.8+5.8.11+8.11.14+\&c.\ldots+(3n-1)(3n+2)(3n+5)$,

and $\Sigma=\quad 2.5.8\ +5.8.11+\&c.\ldots\ +(3n-4)(3n-1)(3n+2)$
$\qquad +(3n-1)(3n+2)(3n+5)$.

$\therefore$ subtracting, $0=2.5.8+9.5.8+9.8.11+\&c.\ldots+9(3n-1)(3n+2)$
$\qquad -(3n-1)(3n+2)(3n+5)$.

$i.e.\ 0=9\{2.5+5.8+\&c.\ldots+(3n-1)(3n+2)\}-10-(3n-1)(3n+2)(3n+5)$.

$$\therefore S_n=\frac{1}{9}\{(3n-1)(3n+2)(3n+5)+10\}.$$

5. $\Sigma = 1 \,.\, 3 \,.\, 5 \,.\, 7 + 3 \,.\, 5 \,.\, 7 \,.\, 9 + \&c.\dots + (2n-1)(2n+1)(2n+3)(2n+5),$

and $\Sigma = 1 \,.\, 3 \,.\, 5 \,.\, 7 + \&c.\dots + (2n-3)(2n-1)(2n+1)(2n+3)$
$+ (2n-1)(2n+1)(2n+3)(2n+5).$

$\therefore\ 0 = 1 \,.\, 3 \,.\, 5 \,.\, 7 + 8 \,.\, 3 \,.\, 5 \,.\, 7 + \&c.\dots + 8(2n-1)(2n+1)(2n+3)$
$- (2n-1)(2n+1)(2n+3)(2n+5).$

i.e. $0 = 8\{1 \,.\, 3 \,.\, 5 + 3 \,.\, 5 \,.\, 7 + \&c.\dots + (2n-1)(2n+1)(2n+3)\}$
$- 15 - (2n-1)(2n+1)(2n+3)(2n+5).$

$$\therefore\ S_n = \frac{1}{8}\{(2n-1)(2n+1)(2n+3)(2n+5) + 15\}.$$

6. $\Sigma = 2 \,.\, 7 \,.\, 12 \,.\, 17 + 7 \,.\, 12 \,.\, 17 \,.\, 22 + \&c.\dots$
$+ (5n-3)(5n+2)(5n+7)(5n+12),$

and $\Sigma = 2 \,.\, 7 \,.\, 12 \,.\, 17 + \&c.\dots + (5n-8)(5n-3)(5n+2)(5n+7)$
$+ (5n-3)(5n+2)(5n+7)(5n+12).$

$\therefore\ 0 = 2 \,.\, 7 \,.\, 12 \,.\, 17 + 20 \,.\, 7 \,.\, 12 \,.\, 17 + \&c.\dots + 20(5n-3)(5n+2)(5n+7)$
$- (5n-3)(5n+2)(5n+7)(5n+12).$

i.e. $0 = 20\{2 \,.\, 7 \,.\, 12 + 7 \,.\, 12 \,.\, 17 + \&c.\dots + (5n-3)(5n+2)(5n+7)\}$
$- 504 - (5n-3)(5n+2)(5n+7)(5n+12).$

$$\therefore\ S_n = \frac{1}{20}\{(5n-3)(5n+2)(5n+7)(5n+12) + 504\}.$$

7. $\Sigma = 1 + \frac{1}{3} + \frac{1}{5} + \&c.\dots + \frac{1}{2n-1} + \frac{1}{2n+1},$

and $\Sigma = 1 + \frac{1}{3} + \&c.\dots + \frac{1}{2n-3} + \frac{1}{2n-1} + \frac{1}{2n+1}.$

$$\therefore\ 0 = 1 - \frac{2}{1 \,.\, 3} - \frac{2}{3 \,.\, 5} - \&c.\dots - \frac{2}{(2n-3)(2n-1)} - \frac{2}{(2n-1)(2n+1)} - \frac{1}{2n+1}.$$

$$i.e.,\ 0 = -2\left\{\frac{1}{1 \,.\, 3} + \frac{1}{3 \,.\, 5} + \&c.\dots + \frac{1}{(2n-1)(2n+1)}\right\} + 1 - \frac{1}{2n+1}.$$

$$\therefore\ S_n = \frac{1}{2}\left\{1 - \frac{1}{2n+1}\right\}.$$

If n is infinite, $\frac{1}{2n+1}$ is zero; $\therefore$ sum to infinity $= \frac{1}{2}$.

8. $\Sigma = \frac{1}{2} + \frac{1}{5} + \frac{1}{8} + \&c.\dots + \frac{1}{3n-1} + \frac{1}{3n+2},$

and $\Sigma = \frac{1}{2} + \frac{1}{5} + \&c.\dots + \frac{1}{3n-4} + \frac{1}{3n-1} + \frac{1}{3n+2}.$

$$\therefore\ 0 = \frac{1}{2} - \frac{3}{2 \,.\, 5} - \frac{3}{5 \,.\, 8} - \&c.\dots - \frac{3}{(3n-4)(3n-1)} - \frac{3}{(3n-1)(3n+2)} - \frac{1}{3n+2}.$$

$$i.e.\ 0 = -3\left\{\frac{1}{2\,.\,5}+\frac{1}{5\,.\,8}+\&c.\dots+\frac{1}{(3n-1)(3n+2)}\right\}+\frac{1}{2}-\frac{1}{3n+2}.$$

$$\therefore\ S_n=\frac{1}{3}\left\{\frac{1}{2}-\frac{1}{3n+2}\right\}.$$

If n is infinite, $\frac{1}{3n+2}$ is zero; $\therefore$ sum to infinity is $\frac{1}{6}$.

9.
$$\frac{1}{1\,.\,3\,.\,5}=\frac{1}{4}\left\{\frac{1}{1\,.\,3}-\frac{1}{3\,.\,5}\right\},$$
$$\frac{1}{3\,.\,5\,.\,7}=\frac{1}{4}\left\{\frac{1}{3\,.\,5}-\frac{1}{5\,.\,7}\right\},$$
$$\dots\dots = \dots\dots\dots$$
$$\frac{1}{(2n-3)(2n-1)(2n+1)}=\frac{1}{4}\left\{\frac{1}{(2n-3)(2n-1)}-\frac{1}{(2n-1)(2n+1)}\right\},$$
$$\frac{1}{(2n-1)(2n+1)(2n+3)}=\frac{1}{4}\left\{\frac{1}{(2n-1)(2n+1)}-\frac{1}{(2n+1)(2n+3)}\right\}.$$

Hence
$$S_n=\frac{1}{4}\left\{\frac{1}{1\,.\,3}-\frac{1}{(2n+1)(2n+3)}\right\},$$

and
$$S_\infty=\frac{1}{4}\times\frac{1}{1\,.\,3}=\frac{1}{12}.$$

10.
$$\frac{1}{2\,.\,5\,.\,8}=\frac{1}{6}\left\{\frac{1}{2\,.\,5}-\frac{1}{5\,.\,8}\right\},$$
$$\frac{1}{5\,.\,8\,.\,11}=\frac{1}{6}\left\{\frac{1}{5\,.\,8}-\frac{1}{8\,.\,11}\right\},$$
$$\dots\dots\dots = \dots\dots\dots$$
$$\frac{1}{(3n-4)(3n-1)(3n+2)}=\frac{1}{6}\left\{\frac{1}{(3n-4)(3n-1)}-\frac{1}{(3n-1)(3n+2)}\right\},$$
$$\frac{1}{(3n-1)(3n+2)(3n+5)}=\frac{1}{6}\left\{\frac{1}{(3n-1)(3n+2)}-\frac{1}{(3n+2)(3n+5)}\right\}.$$

Hence
$$S_n=\frac{1}{6}\left\{\frac{1}{2\,.\,5}-\frac{1}{(3n+2)(3n+5)}\right\},$$
$$S_\infty=\frac{1}{6\,.\,2\,.\,5}=\frac{1}{60}.$$

11.
$$S_n=\left(\frac{1}{x+1}-\frac{1}{x+2}\right)+\left(\frac{1}{x+2}-\frac{1}{x+3}\right)+\dots$$
$$+\left(\frac{1}{x+n-1}-\frac{1}{x+n}\right)+\left(\frac{1}{x+n}-\frac{1}{x+n+1}\right)=\frac{1}{x+1}-\frac{1}{x+n+1}.$$
$$S_\infty=\frac{1}{x+1}.$$

12. $$x\,.\,S_n=\left(\frac{1}{1+x}-\frac{1}{1+2x}\right)+\left(\frac{1}{1+2x}-\frac{1}{1+3x}\right)$$
$$+\ldots+\left\{\frac{1}{1+(n-1)\,x}-\frac{1}{1+nx}\right\}+\left\{\frac{1}{1+nx}-\frac{1}{1+(n+1)\,x}\right\}$$
$$=\frac{1}{1+x}-\frac{1}{1+(n+1)\,x}.$$
$$x\,.\,S_\infty=\frac{1}{1+x}.$$

13. Since $$1+2+3+\ldots+r=\frac{1}{2}r\,(r+1).$$
$$S_n=\frac{2}{1\,.\,2}+\frac{2}{2\,.\,3}+\frac{2}{3\,.\,4}+\ldots=2\left(1-\frac{1}{n+1}\right)$$ [Ex. 5, Art. 232].

14. $1^2+3^2+5^2+\ldots$ to n terms

$=1^2+2^2+3^2+4^2+5^2+6^2+$&c.... to $2n$ terms

$\quad-2^2\qquad-4^2\qquad-6^2-$&c.... to n terms

$=1^2+2^2+3^2+$&c.... to $2n$ terms $-\,2^2(1+2^2+3^2+$&c.... to n terms)

$$=^{*}\ \frac{1}{6}\,2n\,(2n+1)\,(4n+1)-\frac{4}{6}\,n\,(n+1)\,(2n+1)$$
$$=\frac{1}{3}\,n\,(2n+1)\,\{(4n+1)-2\,(n+1)\}=\text{\&c.}$$

15. $1^3+3^3+5^3+$&c.... to n terms

$=1^3+2^3+3^3+4^3+5^3+6^3+$&c.... to $2n$ terms

$\quad-2^3\qquad-4^3\qquad-6^3-$&c.... to n terms

$=1^3+2^3+3^3+$&c.... to $2n$ terms

$\quad-\,2^3\,(1^3+2^3+3^3+$&c.... to n terms)

$$=^{\dagger}\ n^2\,(2n+1)^2-2^3\left\{\frac{1}{4}n^2\,(n+1)^2\right\}$$
$$=n^2\,\{(2n+1)^2-2\,(n+1)^2\}=n^2\,(2n^2-1).$$

16. $$a\,(a+b)=\frac{1}{3b}\,\{a\,(a+b)\,(a+2b)-(a-b)\,a\,(a+b)\}$$
$$(a+b)\,(a+2b)=\frac{1}{3b}\,\{(a+b)\,(a+2b)\,(a+3b)-a\,(a+b)\,(a+2b)\}$$
$$(a+2b)\,(a+3b)=\frac{1}{3b}\,\{(a+2b)\,(a+3b)\,(a+4b)-(a+b)\,(a+2b)\,(a+3b)\}$$
$$\ldots\ldots\ldots\ldots\ldots\ldots=\ldots\ldots\ldots\ldots\ldots\ldots$$
$$(a+\overline{n-1}\,b)\,(a+nb)=\frac{1}{3b}\,\{(a+\overline{n-1}\,b)\,(a+nb)\,(a+\overline{n+1}\,b)-(a+\overline{n-2}\,b)\,(a+\overline{n-1}\,b)\,(a+nb)\}.$$

Hence $$S_n=\frac{1}{3b}\,\{(a+\overline{n-1}\,b)\,(a+nb)\,(a+\overline{n+1}\,b)-(a-b)\,a\,(a+b)\}.$$

---

* § 232, Ex. 3. † § 232, Ex. 4.

MISCELLANEOUS EXAMPLES. VI. Pages 296—300.

A. 1. $(y+3)(y^2-1)-(y-3)(y^2-1)+3(y-1)(y^2-9)-3(y+1)(y^2-9)$
$=(y^2-1)\{(y+3)-(y-3)\}+3(y^2-9)\{(y-1)-(y+1)\}$
$=6(y^2-1)-6(y^2-9)=48.$

2. $$\frac{\{(2y-x)-(2x-y)\}\{(2y-x)^2+(2y-x)(2x-y)+(2x-y)^2\}}{3(y-x)}$$
$$+\frac{\{(2y-x)+(2x-y)\}\{(2y-x)^2-(2y-x)(2x-y)+(2x-y)^2\}}{(x+y)}$$
$$=\{(2y-x)^2+(2y-x)(2x-y)+(2x-y)^2\}$$
$$+\{(2y-x)^2-(2y-x)(2x-y)+(2x-y)^2\}$$
$$=2(2y-x)^2+2(2x-y)^2=\&c.$$

3.
$$\begin{array}{l}
3x^3-13x^2+23x-21\ )\ 6x^3+x^2-44x+21\ (\ 2 \\
\qquad\qquad\qquad\qquad\quad 6x^3-26x^2+46x-42 \\
\qquad\qquad\qquad\qquad\quad \overline{\quad 9\ )\ 27x^2-90x+63} \\
\qquad\qquad\qquad\qquad\quad \overline{3x^2-10x+7}\ )\ 3x^3-13x^2+23x-21\ (\ x-1 \\
\qquad\qquad\qquad\qquad\qquad\qquad\qquad\quad 3x^3-10x^2+7x \\
\qquad\qquad\qquad\qquad\qquad\qquad\qquad\quad \overline{\quad -3x^2+16x-21} \\
\qquad\qquad\qquad\qquad\qquad\qquad\qquad\quad \quad -3x^2+10x-7 \\
\qquad\qquad\qquad\qquad\qquad\qquad\qquad\quad \overline{\qquad\quad 2\ )\ 6x-14} \\
\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \overline{3x-7}\ )\ 3x^2-10x+7\ (\ x-1 \\
\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\quad 3x^2-7x \\
\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\quad \overline{\quad -3x+7} \\
\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\quad \quad -3x+7
\end{array}$$

$3x-7$ is the H.C.F.; thus both expressions vanish if $3x-7=0$.

4. (i) $3x+\frac{2}{y}-1=12x+\frac{5}{y}+14$; whence $3x+\frac{1}{y}=-5$.

Also $3x+\frac{2}{y}-1=\frac{1}{y}-2x-14$; $\therefore 5x+\frac{1}{y}=-13$.

Whence $2x=-8$; $\therefore x=-4$; and then $\frac{1}{y}=7$.

(ii) $\{x-(a+b+c)\}(x-c)=0$; $\therefore$ &c.

5. Let £x be the price he paid for each ox, and £y for each sheep.

Then $$9x+20y=230.$$

Also $$\frac{125}{100}x\times 9+\frac{80}{100}y\times 20=265,\ \&c.$$

6. The equation with the given roots is
$$\left(x-\frac{\alpha+\beta}{\alpha}\right)\left(x-\frac{\alpha+\beta}{\beta}\right)=0,\ \text{ or }\ (\alpha x-\overline{\alpha+\beta})(\beta x-\overline{\alpha+\beta})=0,$$
i.e. $\alpha\beta x^2-(\alpha+\beta)^2x+(\alpha+\beta)^2=0$, *i.e.* $3x^2-16x+16=0$,
since $\alpha\beta=3$ and $\alpha+\beta=-4$.

7.
$$\begin{array}{l}
a^{\frac{2}{3}}b^{-1}-2a^{\frac{1}{3}}b^{-\frac{1}{2}}+4-8a^{-\frac{1}{3}}b^{\frac{1}{2}}+16a^{-\frac{2}{3}}b \\
a^{\frac{1}{3}} \quad +2b^{\frac{1}{2}} \\
\hline
ab^{-1} \quad -2a^{\frac{2}{3}}b^{-\frac{1}{2}}+4a^{\frac{1}{3}}-8b^{\frac{1}{2}}+16a^{-\frac{1}{3}}b \\
\qquad\quad +2a^{\frac{2}{3}}b^{-\frac{1}{2}}-4a^{\frac{1}{3}}+8b^{\frac{1}{2}}-16a^{-\frac{1}{3}}b+32a^{-\frac{2}{3}}b^{\frac{3}{2}} \\
\hline
ab^{-1} \qquad\qquad\qquad\qquad\qquad\qquad\qquad +32a^{-\frac{2}{3}}b^{\frac{3}{2}}
\end{array}$$

8. (i) $\dfrac{(5+4\sqrt{3})(5+4\sqrt{3})}{(5-4\sqrt{3})(5+4\sqrt{3})}=\dfrac{25+40\sqrt{3}+48}{25-48}=$&c.

(ii) Put $\sqrt{25+4\sqrt{34}}=\sqrt{x}+\sqrt{y}$. Square; then $x+y=25$, and $2\sqrt{xy}=4\sqrt{34}$, whence $xy=136$. Hence $x=8$, $y=17$.

9. The series is a G.P. in which $a=5, r=-\frac{3}{4}, n=8$.

Thus
$$S=\frac{a(1-r^n)}{1-r}=\frac{5\left\{1-\left(\frac{3}{4}\right)^8\right\}}{\frac{7}{4}}=\text{\&c.}$$

Sum to infinity $=\dfrac{a}{1-r}=\dfrac{5}{1+\frac{3}{4}}=$&c.

10. Find the A.P. of which 100 is the first term, and 300 the 22nd.

$a=100$, $a+21d=300$; whence $21d=200$, $\therefore\ d=9\frac{11}{21}$.

Thus the A.P. is $100,\ 109\frac{11}{21},\ 119\frac{1}{21}$, &c.... $290\frac{10}{21}$, 300.

The sum of the 20 inserted terms is
$$S=\frac{n}{2}(a+l)=\frac{20}{2}\left(109\tfrac{11}{21}+290\tfrac{10}{21}\right)=4000.$$

B. 1. $3\{x-2y+2z\}-[4y+2\{x-y+z\}]=3x-6y+6z-[2y+2x+2z]$
$=3x-6y+6z-2x-2y-2z=$&c.

2. (i) $(mx+ny)(nx+my)$.
(ii) $x^2(x-y)+y^2(x-y)=(x^2+y^2)(x-y)$.

3. For, if $x+y=4$, $x-y=(x-y)\dfrac{x+y}{4}=\dfrac{1}{4}(x^2-y^2)$.

4. (i) $a(x+b)+b(x+a)=(x+a)(x+b)$, whence $0=x^2-ab$; $\therefore$ &c.
(ii) $\sqrt{12x-3}+\sqrt{x+2}=-\sqrt{7x-13}$;

$\therefore$ squaring, $(12x-3)+2\sqrt{12x-3}\sqrt{x+2}+(x+2)=7x-13$;

whence $\sqrt{12x-3}\sqrt{x+2}=-3x-6$; $\therefore\ (12x-3)(x+2)=(3x+6)^2$,

whence $x^2-5x-14=0$, &c.

5. $x^2-8x+22=(x-4)^2+6$, and $(x-4)^2$ cannot be a negative quantity; $\therefore$ &c.

6. $a^6+2a^3b^2-2a^3c+b^4-2b^2c+c^2=a^6+2a^3(b^2-c)+(b^2-c)^2=(a^3+b^2-c)^2$.

7. $\frac{a}{b}=\frac{c}{d}=$ (say) λ; $\therefore a=b\lambda,\ c=d\lambda$.

$$\therefore \frac{a^2+b^2}{c^2+d^2}=\frac{b^2\lambda^2+b^2}{d^2\lambda^2+d^2}=\frac{b^2}{d^2}.$$

Also $$\frac{(a+b)^2}{(c+d)^2}=\frac{(b\lambda+b)^2}{(d\lambda+d)^2}=\frac{b^2}{d^2},\ \therefore \text{\&c.}$$

8. (i) This is an A.P. since $-1-5=-7-(-1)$.

$$\therefore S=\frac{20}{2}\{2\times5+19(-6)\}=\text{\&c.}$$

(ii) This is a G. P. since $\frac{1}{2\frac{1}{2}}=\frac{\frac{2}{5}}{1}$.

$$\therefore \text{sum to infinity}=\frac{a}{1-r}=\frac{\frac{5}{2}}{1-\frac{2}{5}}=\frac{25}{6}.$$

9. Sum 7, 14, 21..., 994. $a=7$, $d=7$, $l=994$; also, if n be the number of terms, $994=7+(n-1)7$; whence $n=142$.

$$\therefore S=\frac{n}{2}(a+l)=71(7+994)=71071.$$

10. Let the train be x feet long and let the men walk at the rate of y feet per second. The train goes at the rate of $37\frac{1}{2}$ miles per hour which is equivalent to 55 feet per second. Now in 6 seconds the train must go its own length *plus* the distance walked by the man in 6 seconds.

Hence $$55\times6=x+6y.$$

Also in 4 seconds the train must go its own length *minus* the distance walked by the man in 4 seconds.

Hence $55\times4=x-4y$. Whence $y=11$ and $x=264$.

C. 1. $\frac{\sqrt{(2\frac{1}{2})}}{\sqrt{(2\frac{1}{2})}}-\frac{1\frac{1}{2}-0}{0-1\frac{1}{2}}=1-(-1)=2.$

2.
$$\begin{array}{rrrrrl}
2x^2+3xy-2y^2\ ) & 4x^4 & & -9x^2y^2 & +12xy^3-4y^4 & (\ 2x^2-3xy+2y^2 \\
 & 4x^4 & +6x^3y & -4x^2y^2 & & \\
\hline
 & & -6x^3y & -5x^2y^2 & +12xy^3-4y^4 & \\
 & & -6x^3y & -9x^2y^2 & +6xy^3 & \\
\hline
 & & & 4x^2y^2 & +6xy^3-4y^4 & \\
 & & & 4x^2y^2 & +6xy^3-4y^4 &
\end{array}$$

3. Let x, $x+1$ be the numbers. Then we have to prove that

$$\{x+(x+1)\}^2=4x(x+1)+1.$$

4. $x^2-6ax+9a^2=(x-3a)^2$, $x^2-ax-6a^2=(x-3a)(x+2a)$, $3x^2-12a^2=3(x+2a)(x-2a)$. $\therefore$ L.C.M. $=3(x-3a)^2(x+2a)(x-2a)$.

5. $\left(\dfrac{2}{a-2}-\dfrac{2}{a+2}\right)+\left(\dfrac{1}{a+1}-\dfrac{1}{a-1}\right)=\dfrac{8}{a^2-4}+\dfrac{-2}{a^2-1}=\dfrac{6a^2}{(a^2-1)(a^2-4)}$.

6. Let the train travel at x miles an hour, and let the time occupied be y hours. Then the journey would be accomplished in $\left(y-\dfrac{2}{3}\right)$ hours at the rate of $x+5$ miles per hour, and in $(y+1)$ hours at the rate of $x-5$ miles per hour.

Hence $$x\times y=(x+5)\left(y-\frac{2}{3}\right)=(x-5)(y+1).$$

Whence $$5y-\frac{2}{3}x=\frac{10}{3},\quad -5y+x=5;\quad \therefore\ x=25 \text{ and } y=4.$$

7. $\sqrt{(19+6\sqrt{2})}=\sqrt{(19+2\sqrt{18})}=\sqrt{18}+1=3\sqrt{2}+1$.
Thus we have $2+2\sqrt{2}+\sqrt{2}-3\sqrt{3}-2\sqrt{3}+5\sqrt{3}-(3\sqrt{2}+1)=1$.

8. (i) Cf. § 196.

(ii) Let $\dfrac{a}{b}>\dfrac{c}{d}$, then $ad>bc$,

then $\dfrac{a}{b}-\dfrac{ma+nc}{mb+nd}=\dfrac{n(ad-bc)}{b(mb+nd)}=$ a positive quantity, since $ad>bc$.

$$\therefore\ \frac{a}{b}>\frac{ma+nc}{mb+nd}.$$

Also $$\frac{ma+nc}{mb+nd}-\frac{c}{d}=\frac{m(ad-bc)}{d(mb+nd)}=\text{a positive quantity}.$$

$$\therefore\ \frac{ma+nc}{mb+nd}>\frac{c}{d};\ \text{whence \&c.}$$

9. (i) This is an A.P., since $15-21=9-15$;

$$\therefore\ S=\frac{n}{2}\{2a+(n-1)d\}=\frac{8}{2}\{2\times21+7(-6)\}=0.$$

(ii) This is a G.P., since $\dfrac{2}{5}=\dfrac{\cdot8}{2}$;

$$\therefore\ S=\frac{a}{1-r}=\frac{5}{1-\frac{2}{5}}=\frac{25}{3}.$$

10. If x and y be the numbers, $\dfrac{1}{2}(x+y)=17$, and $\sqrt{xy}=15$;

$\therefore\ x+y=34$ and $xy=225$, whence $x=9$ or 25, and $y=25$ or 9.

D. **1.** $\{x^2+a^2\}\{x^2-a^2\}=x^4-a^4$.

2. (i) $(n+1)^4-n^4=\{(n+1)^2-n^2\}\{(n+1)^2+n^2\}=\{2n+1\}\{2n^2+2n+1\}$.

(ii) This is true if $(a+b)^4-4ab(a+b)^2=a^4+b^4-2a^2b^2$,

i.e., if $(a+b)^2\{(a+b)^2-4ab\}=(a^2-b^2)^2$,

i.e., if $(a+b)^2\{(a-b)^2\}=(a^2-b^2)^2$; which is obvious.

3. $\dfrac{x^2-9x+20}{x^2-6x}\cdot\dfrac{x^2-13x+42}{x^2-5x}=\dfrac{(x-4)(x-5)}{x(x-6)}\cdot\dfrac{(x-6)(x-7)}{x(x-5)}=$&c.

4. (i) $(\mathrm{I}\times5-\mathrm{II}\times3)$; $-\dfrac{46}{y}=138$; $\therefore\ y=-\dfrac{1}{3}$;

whence $\dfrac{3}{x}+15=21$; $\therefore\ x=\dfrac{1}{2}$.

(ii) $16(3x^2-4xy+2y^2)=33(x^2-y^2)$;

$\therefore\ 15x^2-64xy+65y^2=0$, whence $x=\dfrac{5y}{3}$, or $\dfrac{13y}{5}$.

If $x=\dfrac{5y}{3}$, $\dfrac{16y^2}{9}=16$; $\therefore\ y=\pm3$, and then $x=\dfrac{5y}{3}=\pm5$.

And if $x=\dfrac{13y}{5}$, $\dfrac{144y^2}{25}=16$; $\therefore\ y=\pm\dfrac{5}{3}$, and then $x=\dfrac{13y}{5}=\pm\dfrac{13}{3}$.

5. Let x be the digit in the units place, y in the tens place. Then the value of the number is $10y+x$. Hence we have $x^2-y^2=10y+x$. If the digits are inverted the value of the number is $10x+y$; thus

$$10x+y=7(x+y),\text{ whence } x=2y.$$

Then $4y^2-y^2=10y+2y$; $\therefore\ y=4$. And then $x=2y=8$.

6. If α, β are the roots of the given equation, $\alpha+\beta=4$, $\alpha\beta=2$. The required equation is $(x-\alpha^3)(x-\beta^3)=0$, *i.e.* $x^2-(\alpha^3+\beta^3)x+\alpha^3\beta^3=0$. Now $\alpha^3+\beta^3=(\alpha+\beta)^3-3\alpha\beta(\alpha+\beta)=4^3-3\,.\,2\,.\,4=40$, and $\alpha^3\beta^3=(\alpha\beta)^3=8$. Thus the required equation is $x^2-40x+8=0$.

7.

$$\begin{array}{rl} & x^6-2x^2+8+x^{-2}-8x^{-4}+16x^{-6}\ (\ x^3-x^{-1}+4x^{-3} \\ (x^3)^2= & x^6 \\ \hline (x^3-x^{-1})^2= & x^6-2x^2 \qquad +x^{-2} \\ \hline (x^3-x^{-1}+4x^{-3})^2= & x^6-2x^2+8+x^{-2}-8x^{-4}+16x^{-6} \end{array}$$

8. Let $\dfrac{a}{b}=\lambda$, then $\dfrac{c}{d}=\lambda$; $\therefore\ a=b\lambda$, $c=d\lambda$.

$$\therefore\ \frac{a^2+b^2}{c^2+d^2}=\frac{b^2\lambda^2+b^2}{d^2\lambda^2+d^2}=\frac{b^2}{d^2}.$$

Also $\left(\dfrac{a-b}{c-d}\right)^2=\left(\dfrac{b\lambda-b}{d\lambda-d}\right)^2=\dfrac{b^2}{d^2}$, $\therefore$ &c.

9. (i) This is a G. P., since $\dfrac{-\frac{1}{2}}{\frac{1}{3}} = \dfrac{\frac{3}{4}}{-\frac{1}{2}}$.

Hence
$$S = \frac{1}{3}\,\frac{1-\left(-\dfrac{3}{2}\right)^{10}}{1-\left(-\dfrac{3}{2}\right)} = \frac{2}{15}\left\{1-\left(\frac{3}{2}\right)^{10}\right\}.$$

(ii) This is a G. P., since $\dfrac{-6}{9} = \dfrac{4}{-6}$.

Hence sum to infinity $= \dfrac{9}{1-\left(-\dfrac{2}{3}\right)} = \dfrac{27}{5}$.

10. In n groups there will be $1+2+3+\ldots+n$, that is $\frac{1}{2}n(n+1)$, of these odd numbers. Hence the last of them will be

$$2\left\{\frac{1}{2}n(n+1)\right\}-1=n(n+1)-1.$$

Hence the nth group will be the sum of n odd numbers of which the greatest is n^2+n-1. The sum is therefore

$$=\frac{n}{2}\left\{2(n^2+n-1)+(n-1)(-2)\right\}=n^3.$$

E. 1. $(x^2-2xy+y^2)-(x^2+xy+y^2)-\{x^2-2xy+y^2\}=-(x^2+xy+y^2)$.

2.
$$\begin{array}{r|l|l}
2a^2-3ab+4b^2 & 6a^4-a^3b+2a^2b^2+13ab^3+4b^4 & 3a^2+4ab+b^2 \\
 & \underline{6a^4-9a^3b+12a^2b^2} & \\
 & 8a^3b-10a^2b^2+13ab^3+4b^4 & \\
 & \underline{8a^3b-12a^2b^2+16ab^3} & \\
 & 2a^2b^2-3ab^3+4b^4 & \\
 & 2a^2b^2-3ab^3+4b^4 &
\end{array}$$

3. $3a^3-3a^2b+ab^2-b^3=3a^2(a-b)+b^2(a-b)=(3a^2+b^2)(a-b)$

and $4a^2-5ab+b^2=(a-b)(4a-b)$.

$\therefore$ H.C.F. $=a-b$; and L.C.M. $=(3a^2+b^2)(a-b)(4a-b)$.

4.
$$\frac{\left\{\dfrac{4}{x-1}+\dfrac{x-1}{x}\right\}x(x-1)}{\left\{\dfrac{1}{x-1}-\dfrac{1}{x}\right\}x(x-1)}=\frac{4x+(x-1)^2}{x-(x-1)}=(x+1)^2.$$

5. $\alpha+\beta=-m$; $\alpha\beta=n$; $\therefore$ $\alpha^2\beta+\beta^2\alpha=\alpha\beta(\alpha+\beta)=n(-m)$.

6. $$3x^2+5x+3=3\left(x^2+\frac{5}{3}x+1\right)=3\left\{x^2+\frac{5}{3}x+\left(\frac{5}{6}\right)^2+1-\frac{25}{36}\right\}$$
$$=3\left\{\left(x+\frac{5}{6}\right)^2+\frac{11}{36}\right\}.$$

The least possible value of $\left(x+\frac{5}{6}\right)^2$ is 0; thus the least possible value of the given expression is $3\times\frac{11}{36}=\frac{11}{12}$, and the expression has its least value when $x+\frac{5}{6}=0$, *i.e.* when $x=-\frac{5}{6}$.

7. $\frac{a}{b}=\frac{b}{c}=\frac{c}{d}=x$ (say); $\therefore\ a=bx,\ b=cx,\ c=dx.$

Thus $\left(\frac{a-b}{b-c}\right)^3=\left(\frac{bx-cx}{b-c}\right)^3=x^3$; and $\frac{a}{d}=\frac{a}{b}\cdot\frac{b}{c}\cdot\frac{c}{d}=x\,.\,x\,.\,x=x^3.$

8.
$$\begin{array}{rl}
 & 4x^{\frac{4}{3}}-12xy^{\frac{1}{2}}-7x^{\frac{2}{3}}y+24x^{\frac{1}{3}}y^{\frac{3}{2}}+16y^2\ (\ 2x^{\frac{2}{3}}-3x^{\frac{1}{3}}y^{\frac{1}{2}}-4y \\
 & 4x^{\frac{4}{3}} \\
\hline
4x^{\frac{2}{3}}-3x^{\frac{1}{3}}y^{\frac{1}{2}}\ ) & -12xy^{\frac{1}{2}} \\
 & -12xy^{\frac{1}{2}}+\ 9x^{\frac{2}{3}}y \\
\hline
4x^{\frac{2}{3}}-6x^{\frac{1}{3}}y^{\frac{1}{2}}-4y\ ) & -16x^{\frac{2}{3}}y \\
 & -16x^{\frac{2}{3}}y+24x^{\frac{1}{3}}y^{\frac{3}{2}}+16y^2
\end{array}$$

9. (i) This is an A. P., since $-2-(-3)=-1-(-2)$.

$$\therefore\ S=\frac{n}{2}\{2a+(n-1)\,d\}=\frac{n}{2}\{-6+(n-1)\,1\}=\frac{1}{2}n\,(n-7).$$

(ii) This is a G. P., since $\dfrac{-\frac{1}{2}}{1}=\dfrac{\frac{1}{4}}{-\frac{1}{2}}$.

$\therefore$ sum to infinity $=\dfrac{a}{1-r}=\dfrac{1}{1-\left(-\frac{1}{2}\right)}=\dfrac{2}{3}.$

10. Let the trains go at the rates of x and y *yards per second*. Then in 4 seconds they must between them travel $(72+60)$ yards. Hence

$$4x+4y=72+60.$$

They would have travelled $(72+60)$ yards in 3 seconds at the rates of $2x$ and y yards per second respectively. Hence

$$3\times2x+3y=72+60.\quad \text{Whence } x=11,\ y=22.$$

And 11 yards a second $=\dfrac{11\times60\times60}{1760}$ miles per hour.

F. 1. $$\{(x+y+z)-(-x+y+z)\}\{(x+y+z)+(-x+y+z)\}$$
$$+\{(x-y+z)-(x+y-z)\}\{(x-y+z)+(x+y-z)\}$$
$$=2x\times 2(y+z)+2(-y+z)\times 2x=8xz.$$

2. $(a-b)^3-c^3\div(a-b)-c=(a-b)^2+(a-b)c+c^2.$

3. For, $y-z=b-c,\ z-x=c-a,\ x-y=a-b;$
$$\therefore\ (y-z)^2+(z-x)^2+(x-y)^2=(b-c)^2+(c-a)^2+(a-b)^2;$$
i.e. $2(x^2+y^2+z^2-yz-zx-xy)=2(a^2+b^2+c^2-bc-ca-ab).$

4. $$\frac{(a+2x)-2a-(a-2x)}{(a-2x)(a+2x)}=\frac{-2(a-2x)}{(a-2x)(a+2x)}=-\frac{2}{a+2x}.$$

5. (i) Eliminate y from I and II, and from II and III; then
$$10x+7z=58,\ \text{and}\ 14x+z=46.$$
Whence $88x=264$; $\therefore\ x=3$. Then $z=4$, and then $y=7$.

(ii) Simplifying each side, $\dfrac{-1}{(x-2)(x-3)}=\dfrac{-1}{(x-6)(x-7)}$;
$$\therefore\ (x-2)(x-3)=(x-6)(x-7),\ \text{whence}\ 8x=36.$$

(iii) The first equation is equivalent to $\dfrac{2(x^2+y^2)}{x^2-y^2}=\dfrac{5}{2}$;
whence $x^2=9y^2$. Then, from $x^2-9y^2=0$, and $x^2+y^2=90$, we have
$$x=\pm 9\ \text{and}\ y=\pm 3.$$

6. Extract the square root by the usual method; thus we shall find that the given expression $=(x^3-2x^2+5x-6)^2$.

Thus, if any integer be put for x in the given expression, the result will be the square of the number obtained by substituting the *same* value of x in the expression x^3-2x^2+5x-6.

7. $$\frac{1}{\sqrt{(12-\sqrt{140})}}=\frac{1}{\sqrt{(12-2\sqrt{35})}}=\frac{1}{\sqrt{7}-\sqrt{5}}$$
$$=\frac{1}{2}(\sqrt{7}+\sqrt{5})=\frac{1}{2}(2{\cdot}64575\ldots+2{\cdot}23607\ldots).$$

8. $$\frac{a-x}{a+x}\gtrless\frac{a^2-x^2}{a^2+x^2},$$

according as $(a-x)(a^2+x^2)\gtrless(a+x)(a^2-x^2)$,

i.e. …………… $a^3-a^2x+ax^2-x^3\gtrless a^3+a^2x-ax^2-x^3$,

………………… $2ax^2\gtrless 2a^2x$,

………………… $x\gtrless a$.

9. $\frac{1}{x}+\frac{1}{y}=\frac{m}{x+y}$; *i.e.* $\frac{x+y}{xy}=\frac{m}{x+y}$;

$\therefore\ x^2+2xy+y^2=mxy;\ \therefore\ x^2+y^2=(m-2)\,xy,\ \therefore\ x^2+y^2\propto xy.$

10. Let the first term of the G.P. be α, and the common ratio r;

then $$a=\frac{\alpha(1-r^n)}{1-r},\ b=\frac{\alpha\,(1-r^{2n})}{1-r},\ c=\frac{\alpha\,(1-r^{3n})}{1-r}.$$

Then $$a^2+b^2=\frac{\alpha^2}{(1-r)^2}\{(1-r^n)^2+(1-r^{2n})^2\}$$
$$=\frac{\alpha^2}{(1-r)^2}(1-r^n)\{1-r^n+(1-r^n)(1+r^n)^2\}$$
$$=\frac{\alpha^2}{(1-r)^2}(1-r^n)\{1-r^{2n}+1-r^{3n}\}$$
$$=a\,(b+c).$$

LXX. PAGES 308—310.

1. 15 . 14 . 13. **2.** 10 . 9 . 8 . 7. **3.** $\underline{|8}$. **4.** ${}_5P_5=\underline{|5}$.

5. ${}_6P_6+{}_6P_5+{}_6P_4+{}_6P_3+{}_6P_2+{}_6P_1$
$=6.5.4.3.2.1+6.5.4.3.2+6.5.4.3+6.5.4+6.5+6=$&c.

6. The 10 boys could be put in a row without any restriction on the two particular boys in $\underline{|10}$ ways. If the two boys are together in either order, the whole can be arranged in $\underline{|9}$ ways; hence the total number of arrangements in which the two particular boys are together is $\underline{|9}\times 2$. Hence the number of arrangements in which the two boys are *not* together is $\underline{|10}-2\,\underline{|9}=8\,\underline{|9}$.

7. $\underline{|n}=720$. Now $\underline{|1}$, $\underline{|2}$, $\underline{|3}$,... are in order 1, 2, 6, 24, 120, 720, &c. Thus $n=6$.

8. $\frac{\underline{|7}}{\underline{|3}\,\underline{|2}}$, $\frac{\underline{|11}}{\underline{|4}\,\underline{|4}\,\underline{|2}}$, $\frac{\underline{|9}}{\underline{|2}}$. **9.** $\frac{\underline{|9}}{\underline{|4}\,\underline{|3}\,\underline{|2}}$. **10.** $\frac{\underline{|7}}{\underline{|3}\,\underline{|2}\,\underline{|2}}$.

11. $\frac{\underline{|8}}{\underline{|3}\,\underline{|3}}$. With e first and s last, there are the 6 letters e, e, s, s, n, c free, and the number of permutations of these is $\frac{\underline{|6}}{\underline{|2}\,\underline{|2}}$.

12. If ${}_nP_5={}_nP_4\times 2$; $n\,(n-1)\,(n-2)\,(n-3)\,(n-4)$
$=n\,(n-1)\,(n-2)\,(n-3)\times 2$. Hence $n-4=2$.

13. $_nP_6 = {}_nP_4\,(n-4)(n-5)$. Hence $(n-4)(n-5)=12$; $n=8$ [the value $n=1$ being inapplicable].

14. $_{16}C_4 = \frac{16\,.\,15\,.\,14\,.\,13}{1\,.\,2\,.\,3\,.\,4}$, $_{15}C_{12} = {}_{15}C_3 = \frac{15\,.\,14\,.\,13}{1\,.\,2\,.\,3}$, $_{20}C_{18} = {}_{20}C_2 = \frac{20\,.\,19}{1\,.\,2}$.

15. $_nC_r = {}_nC_{n-r}$; *also if* $_nC_r = {}_nC_s$; *then either* $s=r$ *or* $s=n-r$. Hence, in the present case $10=n-5$.

Thus $n=15$, and $_nC_2 = \frac{15\,.\,14}{1\,.\,2}$.

16. As in 15, $11=n-10$; $\therefore\ n=21$. Then $_nC_3 = \frac{21\,.\,20\,.\,19}{1\,.\,2\,.\,3}$.

17. The total number of permutations is $_5P_3$; but we must reject all those in which 0 stand first, and these are $_4P_2$ in number. Hence the required number $= {}_5P_3 - {}_4P_2 = 60-12=48$.

18. $\frac{2n(2n-1)(2n-2)}{1\,.\,2\,.\,3} = 12 \times \frac{n\,.\,n-1}{1\,.\,2}$; $\therefore\ 2n-1=9$, [rejecting the inapplicable solutions $n(n-1)=0$].

19. $$\frac{2n(2n-1)(2n-2)}{1\,.\,2\,.\,3} = 24 \times \frac{n(n-1)(n-2)(n-3)}{1\,.\,2\,.\,3\,.\,4};$$
$$\therefore\ 2(2n-1) = 3(n-2)(n-3);$$
whence $n=5$ [rejecting the value $n=\frac{4}{3}$ which is inapplicable].

20. From the total number of permutations, namely $_5P_5$, we must subtract all the permutations beginning with 1 (in number $_4P_4$) and also all those beginning with 21 (in number $_3P_3$). Hence of the numbers greater than 23000 there are $_5P_5 - {}_4P_4 - {}_3P_3$.

21. Each letter can be put in any one of the five boxes, and therefore the whole number of ways is $5 \times 5 \times 5$.

22. We may use 0, 1 or 2 pennies; thus there are 3 ways of choosing from the pennies. There are four ways of choosing from the shillings, and 5 ways of choosing from the pounds. Hence the total number of ways of choosing (and each different choice will clearly give a different sum of money) is $3 \times 4 \times 5$. We have, however, in the above included the case in which no coin at all is taken. Thus the total number of different sums of money is $3 \times 4 \times 5 - 1 = 59$.

23. As in 22, the number of sums is $(4+1)(6+1)(5+1)-1$.

24. Suppose there are a, b together with n other things. Then the total number of combinations of these $r+1$ together consists of (1) the combinations in which a and b both occur, $_nC_{r-1}$ in number; (2) the combinations in which a alone occurs, and in which b alone occurs, $_nC_r \times 2$

in number; (3) the combinations in which neither a nor b occurs, ${}_nC_{r+1}$ in number. Hence

$$ {}_{n+2}C_{r+1} = {}_nC_{r+1} + 2\,{}_nC_r + {}_nC_{r-1}. $$

[The general theorem of which the above is a particular case is proved in *Treatise on Algebra*, Art. 248.]

25. All the things occur equally often, and 4 of the 8 things are taken in every combination. Hence each thing occurs 4 times out of every 8.

26, 27. All the things occur equally often, and n of the $3n$ things are taken in every combination. Hence each thing occurs on the whole once in every three times.

28. ${}_{10}C_6$.

29. An elector can vote for one candidate in ${}_5C_1$ ways, for two candidates in ${}_5C_2$ ways, and for three candidates in ${}_5C_3$ ways. Hence the whole number of ways of voting is ${}_5C_1 + {}_5C_2 + {}_5C_3$.

30. There are ${}_{80}C_3$ different sets of 3 men and ${}_3C_1$ different sets of 1 officer; also either of the first can be taken with either of the second. Hence the number required $= {}_{80}C_3 \times {}_3C_1$.

31. (i) ${}_{30}C_{11}$. (ii) Having chosen one eleven, there are ${}_{19}C_{11}$ ways of choosing another. But the match is the same in whatever order the two elevens are chosen. Hence we have ${}_{30}C_{11} \times {}_{19}C_{11} \times \frac{1}{2} = \frac{\lfloor 30}{2 \lfloor 11 \lfloor 11 \lfloor 8}$.

32. ${}_5C_2$ sets of 2 boys can be taken, and ${}_4C_2$ sets of 2 girls; also each set of boys can be taken with each set of girls, giving ${}_5C_2 \times {}_4C_2 = 10 \times 6$.

33. ${}_4C_2$ sets of 2 ladies, and ${}_6C_2$ sets of two men can be taken; thus there are ${}_4C_2 \times {}_6C_2$ sets of players.

34. Any particular child can be supposed fixed; then the others can be arranged in $\lfloor 7$ different ways.

35. There are as many straight lines as there are pairs of points, that is

$$ {}_nC_2 = \frac{1}{2} n (n-1). $$

Every three straight lines make a triangle. Hence the number of triangles is

$$ \left\{\frac{1}{2} n (n-1)\right\} \left\{\frac{1}{2} n (n-1) - 1\right\} \left\{\frac{1}{2} n (n-1) - 2\right\} \div \lfloor 3. $$

36. There must be tickets for a journey from each of the 30 stations to each of the 29 other stations; ∴ number $= 30 \times 29 = 870$.

37. No two different sets of four coins will be worth the same sum of money. Hence the number of different sums of money is ${}_7C_4 = \frac{7 \cdot 6 \cdot 5 \cdot 4}{1 \cdot 2 \cdot 3 \cdot 4} = 35$.

When two additional sovereigns are added, the number of different sums of money is the above 35, together with the number when *two* sovereigns are taken which is ${}_6C_2$, and the number when *three* sovereigns are taken which is ${}_6C_1$. Thus the whole number of different sums in this case is

$$35+15+6=56.$$

38. There are $n(n-1)$ terms of the type aab, for two of each of the n letters can be taken with any one of the remaining $n-1$ letters.

Also, there are ${}_nC_3$ terms of the type abc. Hence the whole number of different products of three dimensions

$$=n(n-1)+\frac{1}{6}n(n-1)(n-2)$$

$$=\frac{1}{6}n(n-1)(n+4).$$

39. Each of the 6 flags can be put at the top, and then each of the 6 flags can be put next, and so on. Hence the whole number

$$=6\times 6\times 6\times 6=1296.$$

40. The total number of arrangements of the n men without restrictions is $\underline{|n}$.

There are $\underline{|n-1}$ arrangements with a at the top and the same number with a at the bottom. Similarly for b. But there are $2\,\underline{|n-2}$ arrangements with a at one end and b at the other. Hence there are $4\,\underline{|n-1}-2\,\underline{|n-2}$ arrangements with either a or b at one end.

Thus the number required

$$=\underline{|n}-4\,\underline{|n-1}+2\,\underline{|n-2}$$

$$=\underline{|n-2}\,\{n(n-1)-4(n-1)+2\}$$

$$=(n-2)(n-3)\,\underline{|n-2}.$$

LXXI. Pages 317, 318.

1. $$x^6+6ax^5+\frac{6.5}{1.2}.a^2x^4+\frac{6.5.4}{1.2.3}a^3x^3+\frac{6.5.4.3}{1.2.3.4}a^4x^2$$
$$+\frac{6.5.4.3.2}{1.2.3.4.5}a^5x+a^6=\&c.$$

2. $$\{1+(-x)^2\}^5=1+5(-x^2)+\frac{5\,.\,4}{1\,.\,2}(-x^2)^2+\frac{5\,.\,4\,.\,3}{1\,.\,2\,.\,3}(-x^2)^3+\frac{5\,.\,4\,.\,3\,.\,2}{1\,.\,2\,.\,3\,.\,4}(-x^2)^4+(-x^2)^5=\&c.$$

3. $$\{(3x)+(-2y)\}^4=(3x)^4+4(3x)^3(-2y)+\frac{4\,.\,3}{1\,.\,2}(3x)^2(-2y)^2+\frac{4\,.\,3\,.\,2}{1\,.\,2\,.\,3}(3x)(-2y)^3+(-2y)^4=\&c.$$

4. $$\{(2a)+(3a^2)\}^4=(2a)^4+4(2a)^3(3a^2)+\frac{4\,.\,3}{1\,.\,2}\,.\,(2a)^2(3a^2)^2+\frac{4\,.\,3\,.\,2}{1\,.\,2\,.\,3}(2a)(3a^2)^3+(3a^2)^4=\&c.$$

5. $$\{(2x^2)+(-1)^6\}=(2x^2)^6+6(2x^2)^5(-1)+\frac{6\,.\,5}{1\,.\,2}\,.\,(2x^2)^4(-1)^2+\&c.=\&c.$$

6. $$\{y+(-x)\}^7=y^7+7y^6(-x)+\frac{7\,.\,6}{1\,.\,2}\,.\,y^5(-x)^2+\&c.=\&c.$$

For **7—12**, cf. § 243.

7. $\frac{10\,.\,9}{1\,.\,2}\,.\,a^8(-3b)^2=\&c.$

8. $\frac{12\,.\,11\,.\,10\,.\,9}{1\,.\,2\,.\,3\,.\,4}(2x)^8(-x^2)^4=\&c.$

9. $\frac{8\,.\,7\,.\,6\,.\,5\,.\,4}{1\,.\,2\,.\,3\,.\,4\,.\,5}(2a)^3\left(-\frac{1}{2}\right)^5=\&c.$

10. ${}_{10}C_6(-x)^6={}_{10}C_4x^6=\&c.$

11. ${}_{20}C_{17}x^{17}={}_{20}C_3x^{17}=\&c.$

12. ${}_{22}C_{20}(-x)^{20}={}_{22}C_2x^{20}=\&c.$

13. $$\left\{a^4+4a^3(\sqrt{b})+\frac{4\,.\,3}{1\,.\,2}a^2(\sqrt{b})^2+\frac{4\,.\,3\,.\,2}{1\,.\,2\,.\,3}a(\sqrt{b})^3+(\sqrt{b})^4\right\}$$
$$+\left\{a^4-4a^3(\sqrt{b})+\frac{4\,.\,3}{1\,.\,2}a^2(\sqrt{b})^2-\frac{4\,.\,3\,.\,2}{1\,.\,2\,.\,3}a(\sqrt{b})^3+(\sqrt{b})^4\right\}$$
$$=2\left\{a^4+\frac{4\,.\,3}{1\,.\,2}a^2(\sqrt{b})^2+(\sqrt{b})^4\right\}=\&c.$$

14. $$\left\{a^6+6a^5(\sqrt{b})+\frac{6\,.\,5}{1\,.\,2}a^4(\sqrt{b})^2+\&c.+(\sqrt{b})^6\right\}$$
$$+\left\{a^6-6a^5(\sqrt{b})-\frac{6\,.\,5}{1\,.\,2}a^4(\sqrt{b})^2+\&c.+(\sqrt{b})^6\right\}$$
$$=2\left\{a^6+\frac{6\,.\,5}{1\,.\,2}\,.\,a^4(\sqrt{b})^2+\frac{6\,.\,5\,.\,4\,.\,3}{1\,.\,2\,.\,3\,.\,4}a^2(\sqrt{b})^4+(\sqrt{b})^6\right\}=\&c.$$

15. As in 13 and 14, we have
$$2\left\{a^8+\frac{8\,.\,7}{1\,.\,2}a^6(\sqrt{b})^2+\frac{8\,.\,7\,.\,6\,.\,5}{1\,.\,2\,.\,3\,.\,4}a^4(\sqrt{b})^4+\frac{8\,.\,7\,.\,6\,.\,5\,.\,4\,.\,3}{1\,.\,2\,.\,3\,.\,4\,.\,5\,.\,6}a^2(\sqrt{b})^6+(\sqrt{b})^8\right\}=\&c.$$

16. The middle term $=$ 5th term $=\dfrac{8.7.6.5}{1.2.3.4}x^4=\&c.$

17. The middle term $=$ 6th term $=\dfrac{10.9.8.7.6}{1.2.3.4.5}x^5=\&c.$

18. Middle term $=$ 5th term $=\dfrac{8.7.6.5}{1.2.3.4}(2x)^4(-3y)^4=\&c.$

19. The coefficients are ${}_{m+n}C_m$, and ${}_{m+n}C_n$ respectively, which are equal by § 238.

20. $$(x^2)^8+8(x^2)^7\left(\frac{1}{x^2}\right)+\frac{8.7}{1.2}.(x^2)^6\left(\frac{1}{x^2}\right)^2+\frac{8.7.6}{1.2.3}(x^2)^5\left(\frac{1}{x^2}\right)^3$$
$$+\frac{8.7.6.5}{1.2.3.4}(x^2)^4\left(\frac{1}{x^2}\right)^4+\frac{8.7.6.5.4}{1.2.3.4.5}(x^2)^3\left(\frac{1}{x^2}\right)^5$$
$$+\frac{8.7.6.5.4.3}{1.2.3.4.5.6}(x^2)^2\left(\frac{1}{x^2}\right)^6+\frac{8.7.6.5.4.3.2}{1.2.3.4.5.6.7}(x^2)\left(\frac{1}{x^2}\right)^7$$
$$+\left(\frac{1}{x^2}\right)^8=\&c.$$

21. ${}_{15}C_{10}(4x)^5\left(-\dfrac{1}{2\sqrt{x}}\right)^{10}={}_{15}C_5\dfrac{4^5x^5}{2^{10}x^5}={}_{15}C_5=\&c.$

22. ${}_{20}C_{16}(x^2)^4\left(-\dfrac{1}{x}\right)^{16}={}_{20}C_4\dfrac{x^8}{x^{16}}=\&c.$

23. General term is ${}_{2n}C_r(x^2)^{2n-r}\left(-\dfrac{a^3}{x}\right)^r={}_{2n}C_r(-a^3)^r x^{4n-3r}.$

The index of x is n, if $r=n$; thus the coefficient required is

$${}_{2n}C_n(-a^3)^n=\&c.$$

24. There are $2n+2$ terms; thus middle terms are

$$(n+1)\text{th term}={}_{2n+1}C_n x^{n+1}\left(\frac{1}{x}\right)^n=\&c.$$
$$(n+2)\text{th term}={}_{2n+1}C_{n+1} x^n\left(\frac{1}{x}\right)^{n+1}=\&c.$$

25. Greatest value of ${}_8C_r$ is when $r=4$; [§ 240] but ${}_8C_4$ is the coefficient of the fifth term.

26. By § 240, ${}_9C_r={}_9C_{r-1}$ if $r=\frac{1}{2}(9+1)=5$; but ${}_9C_4$ and ${}_9C_5$ are the coefficients of the 5th and 6th terms.

27. By § 248, if the rth term is the greatest,

$$r>\frac{42}{5},\text{ and }<\frac{42}{5}+1;\ \therefore r=9.$$

28. As in 27, $r>\dfrac{114}{11}$, and $<\dfrac{114}{11}+1$; $\therefore\ r=11.$

29. Required the greatest term in the expansion of $3^{12}\left(1+\frac{4x}{3}\right)^{12}$, when $x=\frac{1}{2}$.

As in 27, $r>\frac{26}{5}$, and $<\frac{26}{5}+1$; $\therefore r=6$.

30. The $(r+1)$th coefficient $=r$th coefficient $\times\frac{n-r+1}{r}\times\frac{7}{5}$.

Hence the rth and $(r+1)$th coefficients will be equal, if $(23-r+1)\,7=5r$, or $r=14$.

31. ${}_{n}C_{n-r}\,(ax)^r\,(-b)^{n-r}=\&c.$

32. Term in x^n in first expansion is ${}_{2n}C_n x^n$;

and in second expansion is ${}_{2n-1}C_n\,(-x)^n$;

and
$$ {}_{2n}C_n=\frac{\underline{|2n}}{\underline{|n}\;\underline{|n}}=\frac{2n\,\underline{|2n-1}}{\underline{|n}\;\underline{|n-1}\,.\,n}=2\frac{\underline{|2n-1}}{\underline{|n}\;\underline{|n-1}}=2\,{}_{2n-1}C_n. $$

33. The $(n+1)$th term is ${}_{2n}C_n x^n=\frac{\underline{|2n}}{\underline{|n}\;\underline{|n}}x^n$
$$=\frac{1\,.\,3\,.\,5\ldots(2n-1)}{\underline{|n}}\cdot\frac{2\,.\,4\,.\,6\ldots 2n}{1\,.\,2\,.\,3\ldots n}x^n=\&c.$$

34. $99^4=(100-1)^4=100^4-4\,.\,100^3+6\,.\,100^2-4\,.\,100+1=\&c.$
$999^3=(1000-1)^3=1000^3-3\,.\,1000^2+3\,.\,1000-1=\&c.$

LXXII. Pages 323, 324.

1. $c_1-2c_2+\ldots+(-1)^{n-1}nc_n$
$$=n-2\,.\,\frac{n(n-1)}{1\,.\,2}+3\,.\,\frac{n(n-1)(n-2)}{1\,.\,2\,.\,3}-\&c.+(-1)^{n-1}n\,.\,1$$
$$=n\left\{1-\frac{n-1}{1}+\frac{(n-1)(n-2)}{1\,.\,2}-\&c.+(-1)^{n-1}\right\}=n(1-1)^{n-1}=0.$$

2. From § 249, II, $c_0-c_1+c_2-\ldots+(-1)^n c_n=0$;
from the last example $-c_1+2c_2-\ldots+(-1)^n nc_n=0$.
Add these two equations.

3. $c_0+\frac{1}{2}c_1+\frac{1}{3}c_2+\ldots+\frac{1}{n+1}c_n$
$$=1+\frac{1}{2}n+\frac{1}{3}\,\frac{n(n-1)}{1\,.\,2}+\ldots+\frac{1}{n+1}$$
$$=\frac{1}{n+1}\left\{(n+1)+\frac{(n+1)n}{1\,.\,2}+\frac{(n+1)n(n-1)}{1\,.\,2\,.\,3}+\&c.+1\right\}$$
$$=\frac{1}{n+1}\{-1+(1+1)^{n+1}\}=\frac{2^{n+1}-1}{n+1}.$$

4. $\frac{c_1}{c_0}+2\frac{c_2}{c_1}+3\frac{c_3}{c_2}+\ldots+n\frac{c_n}{c_{n-1}}$

$$=\frac{n}{1}+2\frac{n(n-1)}{1\,.\,2}\cdot\frac{1}{n}+3\frac{n(n-1)(n-2)}{1\,.\,2\,.\,3}\cdot\frac{1\,.\,2}{n(n-1)}+\ldots+n\cdot\frac{1}{n}$$

$$=n+(n-1)+(n-2)+\&c.+1=\frac{1}{2}n(n+1).$$

5. Add the equation in Ex. 3, to that in § 249, Ex. 2.

6. $c_0+c_1x+2c_2x^2+3c_3x^3+\ldots+rc_rx^r\ldots+nc_nx^n$

$$=1+nx+\frac{n(n-1)}{1}x^2+\frac{n(n-1)(n-2)}{1\,.\,2}x^3+\ldots+\frac{\underline{|n}}{\underline{|n-r}\,\underline{|r-1}}x^r\ldots+nx^n$$

$$=1+nx\left\{1+(n-1)x+\frac{(n-1)(n-2)}{1\,.\,2}x^2+\ldots+\frac{\underline{|n-1}}{\underline{|n-r}\,\underline{|r-1}}x^{r-1}\ldots+x^{n-1}\right\}$$

$$=1+nx(1+x)^{n-1}.$$

7. $$(1+x)^n=c_0+c_1x+c_2x^2+\ldots c_{n-1}x^{n-1}+c^nx^n;$$

also $$(1+x)^n=c_n+c_{n-1}x+c_{n-2}x^2+\ldots c_1x^{n-1}+c_0x^n.$$

The coefficient of x^{n-1} in the product of the two series on the right is $c_0c_1+c_1c_2+\ldots+c_{n-1}c_n$.

Hence (§ 272), $c_0c_1+c_1c_2+\&c.$ is the coefficient of x^{n-1} in the expansion of $(1+x)^n\times(1+x)^n$, *i.e.* of $(1+x)^{2n}$; which coefficient is $\frac{\underline{|2n}}{\underline{|n-1}\,\underline{|n+1}}$.

8. Similarly, coefficient of x^{n-2} in the product of the same two series is $c_0c_2+c_1c_3+\ldots+c_{n-2}c_n$; which is thus equal to the coefficient of x^{n-2} in the expansion of $(1+x)^{2n}$, *i.e.* to $\frac{\underline{|2n}}{\underline{|n-2}\,\underline{|n+2}}$.

9. $(1+x)^n=c_n+{}_nc_{n-1}+\ldots\ldots+c_0x^n;$

$(1-x)^n=c_0-c_1x+\ldots\ldots+(-1)^nc_nx^n.$

The coefficient of x^n in the product of these series is

$$c_0^2-c_1^2+\ldots\ldots+(-1)^nc_n^2;$$

which is therefore equal to the coefficient of x^n in $(1-x^2)^n$. In this expansion there are no *odd* powers of x; $\therefore$ if n is odd, coefficient of x^n is 0. If n is even, the coefficient of x^n is

$$\frac{\underline{|n}}{\underline{|\tfrac{1}{2}n}\,\underline{|\tfrac{1}{2}n}}(-1)^{\frac{n}{2}}.$$

10. and **11.**

$$c_0+2c_1x+3c_2x^2+4c_3x^3+\ldots+(n+1)\,c_nx^n$$
$$=c_0+c_1x+c_2x^2+c_3x^3+\ldots+c_nx^n$$
$$+c_1x+2c_2x^2+3c_3x^3+\ldots+nc_nx^n$$
$$=(1+x)^n+nx+2\,.\,\frac{n(n-1)}{1\,.\,2}x^2+3\,.\,\frac{n(n-1)\,.\,(n-2)}{1\,.\,2\,.\,3}x^3+\ldots+nx^n$$
$$=(1+x)^n+nx\,(1+x)^{n-1}.$$

Hence $\quad c_0+2c_1x+3c_2x^2+\ldots+(n+1)\,c_nx^n=(1+x)^n+nx\,(1+x)^{n-1}$

$$c_1+2c_2x+\ldots+n\,.\,c_nx^{n-1}=n\,(1+x)^{n-1}.$$

Also $\quad c_n+c_{n-1}x+c_{n-2}x^2+\ldots+c_1x^{n-1}+c_0x^n=(1+x)^n.$

Hence $c_0{}^2+2c_1{}^2+3c_2{}^2+\ldots+(n+1)\,c_n{}^2=$coefficient of x^n in

$$\{(1+x)^n+nx\,(1+x)^{n-1}\}\times(1+x)^n=\frac{\underline{|2n}}{\underline{|n}\,\underline{|n}}+n\frac{\underline{|2n-1}}{\underline{|n}\,\underline{|n-1}}\,.$$

Also $c_1{}^2+2c_2{}^2+3c_3{}^2+\ldots+nc_n{}^2=$coefficient of x^{n-1} in

$$n\,(1+x)^{n-1}\times(1+x)^n=n\,\frac{\underline{|2n-1}}{\underline{|n}\,\underline{|n-1}}\,.$$

12. We have to prove that

$$n-\frac{1}{2}\,.\,\frac{n(n-1)}{1\,.\,2}+\frac{1}{3}\,.\,\frac{n(n-1)(n-2)}{1\,.\,2\,.\,3}-\ldots+(-1)^n.\frac{1}{n}=\frac{1}{1}+\frac{1}{2}+\frac{1}{3}+\ldots+\frac{1}{n}.$$

The theorem is true when $n=2$, for

$$2-\frac{1}{2}\,.\,\frac{2\,.\,1}{1\,.\,2}=\frac{1}{1}+\frac{1}{2}.$$

Assume that the theorem is true for all integer values up to n. Then it will be true for $n+1$ provided

$$(n+1)-\frac{1}{2}\frac{(n+1)\,n}{1\,.\,2}+\frac{1}{3}\frac{(n+1)\,n\,(n-1)}{1\,.\,2\,.\,3}-\ldots+(-1)^{n+1}\frac{1}{n+1}$$
$$=\frac{1}{1}+\frac{1}{2}+\ldots+\frac{1}{n}+\frac{1}{n+1},$$

that is (by subtraction), if

$$1-\frac{n}{1\,.\,2}+\frac{n(n-1)}{1\,.\,2\,.\,3}-\ldots+(-1)^{n+1}\frac{1}{n+1}=\frac{1}{n+1},$$

or, if $\quad \frac{1}{n+1}\left\{(n+1)-\frac{(n+1)\,n}{1\,.\,2}+\frac{(n+1)\,n\,(n-1)}{1\,.\,2\,.\,3}-\ldots+(-1)^{n+1}\right\}=\frac{1}{n+1},$

that is, if $\quad \frac{1}{n+1}\{1-(1-1)^{n+1}\}=\frac{1}{n+1},$

which is obvious. Thus, *if* the theorem be true for any integral value of n, it will be true for the next greater value. But the theorem is true when $n=2$, and therefore when $n=3$, and therefore when $n=4$, and so on.

13. See *Treatise on Algebra*, Art. 259, Ex. 3, or Art. 297, Ex. 3.

14. Since the coefficients of the terms equidistant respectively from the beginning and the end are equal, every coefficient except the middle one, when there is a middle one, occurs twice; and therefore the sum of the coefficients, exclusive of the middle one, is even. But the sum of all the coefficients is even [Art. 249, I], whence it follows that the coefficient of the middle term, whenever there is one, is even.

LXXIII. Pages 330, 333.

1. $$1+(-4)(-x)+\frac{(-4)(-5)}{1.2}(-x)^2+\frac{(-4)(-5)(-6)}{1.2.3}(-x)^3$$
$$+\frac{(-4)(-5)(-6)(-7)}{1.2.3.4}(-x)^4+\ldots=\&c.$$

2. $$1+(-4).2x+\frac{(-4)(-5)}{1.2}(2x)^2+\frac{(-4)(-5)(-6)}{1.2.3}(2x)^3$$
$$+\frac{(-4)(-5)(-6)(-7)}{1.2.3.4}(2x)^4+\ldots=\&c.$$

3. $$(2-x)^{-3}=2^{-3}\left(1-\frac{x}{2}\right)^{-3}=2^{-3}\left\{1+(-3)\left(-\frac{x}{2}\right)\right\}$$
$$+\frac{(-3)(-4)}{1.2}\left(-\frac{x}{2}\right)^2+\frac{(-3)(-4)(-5)}{1.2.3}\left(-\frac{x}{2}\right)^3$$
$$+\frac{(-3)(-4)(-5)(-6)}{1.2.3.4}\left(-\frac{x}{2}\right)^4+\frac{(-3)(-4)(-5)(-6)(-7)}{1.2.3.4.5}\left(-\frac{x}{2}\right)^5$$
$$=\&c.$$

4. $$1+\left(-\frac{2}{3}\right)(-3x)+\frac{\left(-\frac{2}{3}\right)\left(-\frac{5}{3}\right)}{1.2}(-3x)^2+\frac{\left(-\frac{2}{3}\right)\left(-\frac{5}{3}\right)\left(-\frac{8}{3}\right)}{1.2.3}(-3x)^3$$
$$+\frac{\left(-\frac{2}{3}\right)\left(-\frac{5}{3}\right)\left(-\frac{8}{3}\right)\left(-\frac{11}{3}\right)}{1.2.3.4}(-3x)^4+\ldots=\&c.$$

5. $$1+\frac{3}{5}(-5x)+\frac{\frac{3}{5}\left(-\frac{2}{5}\right)}{1.2}(-5x)^2+\frac{\frac{3}{5}\left(-\frac{2}{5}\right)\left(-\frac{7}{5}\right)}{1.2.3}(-5x)^3$$
$$+\frac{\frac{3}{5}\left(-\frac{2}{5}\right)\left(-\frac{7}{5}\right)\left(-\frac{12}{5}\right)}{1.2.3.4}(-5x)^4=\&c.$$

6. $$\frac{(-5)(-6)(-7)\ldots\ldots(-r-4)}{1.2.3\ldots r}(-x)^r=+\frac{5.6.7\ldots(r+4)}{\underline{|r}}x^r.$$

7. $$\frac{(-n)(-n-1)(-n-2)\dots(-n-r+1)}{1.2\dots r}(-x)^r$$
$$= +\frac{n(n+1)(n+2)\dots(n+r-1)}{\underline{|r}}x^r.$$

8. $$\frac{\left(-\frac{3}{2}\right)\left(-\frac{5}{2}\right)\dots\left(-\frac{2r+1}{2}\right)}{1.2\dots r}(-x)^r = +\frac{3.5.7\dots(2r+1)}{\underline{|r}\,2^r}x^r.$$

9. $$\frac{\frac{3}{2}\left(\frac{1}{2}\right)\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)\dots\left(-\frac{2r-5}{2}\right)}{1.2.3.4\dots r}x^r$$
$$= (-1)^r\frac{3.1.1.3.5\dots(2r-5)}{\underline{|r}.2^r}x^r.$$

10. $$\frac{\left(-\frac{2}{3}\right)\left(-\frac{5}{3}\right)\dots\left(-\frac{3r-1}{3}\right)}{1.2\dots r}x^r = (-1)^r\frac{2.5\dots(3r-1)}{\underline{|r}.3^r}x^r.$$

11. $$\frac{\left(-\frac{5}{2}\right)\left(-\frac{7}{2}\right)\dots\left(-\frac{2r+3}{2}\right)}{1.2\dots r}(-2x)^r = +\frac{5.7\dots(2r+3)}{\underline{|r}}x^r.$$

12. $$\frac{\left(-\frac{4}{3}\right)\left(-\frac{7}{3}\right)\dots\left(-\frac{3r+1}{3}\right)}{1.2\dots r}(3x)^r = (-1)^r\frac{4.7.10\dots(3r+1)}{\underline{|r}}x^r.$$

13. $$\frac{a}{\sqrt{(a^2-x^2)}} = \frac{1}{\sqrt{\left(1-\frac{x^2}{a^2}\right)}} = \left(1-\frac{x^2}{a^2}\right)^{-\frac{1}{2}}.$$

And the coefficient of x^6 is

$$\frac{\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)\left(-\frac{5}{2}\right)}{1.2.3}\left(-\frac{1}{a^2}\right)^3 = \frac{1.3.5}{2.4.6}\frac{1}{a^6}.$$

14. General term is

$$\frac{\frac{2}{3}\left(-\frac{1}{3}\right)\left(-\frac{4}{3}\right)\dots\left(-\frac{3r-5}{3}\right)}{1.2.3\dots r}(-3x)^r = -\frac{2.1.4\dots(3r-5)}{\underline{|r}}x^r.$$

15. $$\{1-(2x-3x^2)\}^{-1} = 1+(2x-3x^2)+(2x-3x^2)^2$$
$$+(2x-3x^2)^3+(2x-3x^2)^4+\dots$$
$$=1+(2x-3x^2)+(4x^2-12x^3+9x^4)$$
$$+(8x^3-36x^4+\dots)+(16x^4-\dots)$$
$$=1+\dots-4x^3-11x^4\dots$$

16. $\{1-x(1-3x)\}^{-\frac{1}{2}}$

$$=1+\frac{1}{2}x(1-3x)+\frac{1\,.\,3}{2\,.\,4}x^2(1-3x)^2$$

$$+\frac{1\,.\,3\,.\,5}{2\,.\,4\,.\,6}x^3(1-3x)^3+\frac{1\,.\,3\,.\,5\,.\,7}{2\,.\,4\,.\,6\,.\,8}x^4(1-3x)^4+\dots$$

Coefficient of x^3 is $\quad \frac{1\,.\,3}{2\,.\,4}(-6)+\frac{1\,.\,3\,.\,5}{2\,.\,4\,.\,6}=\text{\&c.}$

Coefficient of x^4 is $\quad \frac{1\,.\,3}{2\,.\,4}\,.\,9+\frac{1\,.\,3\,.\,5}{2\,.\,4\,.\,6}(-9)+\frac{1\,.\,3\,.\,5\,.\,7}{2\,.\,4\,.\,6\,.\,8}=\text{\&c.}$

17. $$\frac{\frac{7}{2}\,.\,\frac{5}{2}\,.\,\frac{3}{2}\,.\,\frac{1}{2}\left(-\frac{1}{2}\right)}{1\,.\,2\,.\,3\,.\,4\,.\,5}(3x)^5=\text{\&c.}$$

18. $$\frac{\frac{19}{6}\,.\,\frac{13}{6}\,.\,\frac{7}{6}\,.\,\frac{1}{6}\left(-\frac{5}{6}\right)}{1\,.\,2\,.\,3\,.\,4\,.\,5}(3x)^5=\text{\&c.}$$

19. (i) $\sqrt{110}=\sqrt{100\left(1+\frac{1}{10}\right)}=10\left(1+\frac{1}{10}\right)^{\frac{1}{2}}$

$$=10\left\{1+\frac{1}{2}\,.\,\frac{1}{10}-\frac{1\,.\,1}{2\,.\,4}\left(\frac{1}{10}\right)^2+\frac{1\,.\,1\,.\,3}{2\,.\,4\,.\,6}\left(\frac{1}{10}\right)^3\right.$$

$$\left.-\frac{1\,.\,1\,.\,3\,.\,5}{2\,.\,4\,.\,6\,.\,8}\left(\frac{1}{10}\right)^4\right\}$$

$$=10\{1+\cdot05-\cdot00125+\cdot0000625-\cdot0000039\}=\text{\&c.}$$

(ii) $\sqrt[3]{130}=\sqrt[3]{125}\sqrt[3]{\left(1+\frac{1}{25}\right)}$

$$=5\left\{1+\frac{1}{3}(\cdot04)-\frac{1\,.\,2}{3\,.\,6}(\cdot0016)+\frac{1\,.\,2\,.\,5}{3\,.\,6\,.\,9}(\cdot000064)-\dots\right\}$$

$$=5\{1+\cdot01\dot{3}-\cdot0001\dot{7}+\cdot000003\}=\text{\&c.}$$

(iii) $\sqrt[4]{630}=\sqrt[4]{625\left(1+\frac{1}{125}\right)}=5\;(1+\cdot008)^{\frac{1}{4}}$

$$=5\left\{1+\frac{1}{4}\times\cdot008-\frac{1\,.\,3}{4\,.\,8}\times\cdot000064+\dots\right\}$$

$$=5\{1+\cdot002-\cdot000006+\dots\}=\text{\&c.}$$

20. $(1+x+x^2+x^3)^3=\left(\frac{1-x^4}{1-x}\right)^3=(1-x^4)^3\times(1-x)^{-3}$

$$=(1-3x^4+3x^8-x^{12})\times\frac{1}{2}\{1\,.\,2+2\,.\,3x$$

$$+3\,.\,4x^2\dots+5\,.\,6x^4+\dots+7\,.\,8x^6+\dots+9\,.\,10x^8+\dots\}.$$

The coefficient of $x^6 = \frac{1}{2}\{7 . 8 - 3 . 3 . 4\}$.

The coefficient of $x^8 = \frac{1}{2}\{3 . 1 . 2 - 3 . 5 . 6 + 9 . 10\}$.

21. $(1+x+x^2)^{-1} = \left(\frac{1-x^3}{1-x}\right)^{-1} = (1-x)(1-x^3)^{-1} = (1-x)(1+x^3+x^6+\dots)$,
and the coefficients of x^2 and x^5 are clearly zero.

22. $(1+x+x^2+x^3+x^4)^{-3} = \left(\frac{1-x^5}{1-x}\right)^{-3} = (1-x)^3(1-x^5)^{-3}$

$$= (1-3x+3x^2-x^3) \times \tfrac{1}{2}(1 . 2 + 2 . 3x^5 + 3 . 4x^{10} + \dots)$$

and the coefficients of x^4 and x^9 are clearly zero.

23. The term is $\dfrac{\frac{27}{5} . \frac{22}{5} . \frac{17}{5} . \frac{12}{5} . \frac{7}{5} . \frac{2}{5}\left(-\frac{3}{5}\right)}{1 . 2 . 3 . 4 . 5 . 6 . 7} x^7$.

24. The $(r+1)$th term is $\dfrac{5 . 8 . 11 \dots (3r+2)}{3 . 6 . 9 \dots 3r} x^r$.

Hence if $x = \dfrac{15}{17}$, the $(r+1)$th term is $\overset{>}{\underset{<}{=}}$ rth term,

according as $$\frac{3r+2}{3r} . \frac{15}{17} \overset{>}{\underset{<}{=}} 1,$$

i.e. as $$r \overset{<}{\underset{>}{=}} 5.$$

Thus the 5th and 6th terms are equal and are greater than any other terms.

25. The $(r+1)$th term is

$$\frac{13 . 14 \dots (r+12)}{1 . 2 \dots r} x^r.$$

$\therefore$ the $(r+1)$th term is $\overset{>}{\underset{<}{=}}$ rth term,

according as $$\frac{r+12}{r} x \overset{>}{\underset{<}{=}} 1,$$

i.e. as $$\frac{r+12}{r} . \frac{1}{3} \overset{>}{\underset{<}{=}} 1,$$

or as $$r \overset{<}{\underset{>}{=}} 6.$$

Thus the 6th and 7th terms are equal and are greater than any of the other terms.

26. The expansion is

$$1+\frac{12}{5}(-x)+\frac{12.7}{5.10}(-x)^2+\frac{12.7.2}{5.10.15}(-x)^3+\frac{12.7.2(-3)}{5.10.15.20}(-x)^4+\dots.$$

After the 4th term *two* additional negative factors are added to each successive term; the signs of all the terms after the 4th are therefore all negative.

Thus only the first and third terms are positive.

27. The $(r+1)$th term is

$$\frac{4.7.10\dots(3r+1)}{3.6.9\dots 3r}x^r.$$

Hence the $(r+1)$th term is $\gtreqless$ rth term,

according as $$\frac{3r+1}{3r}\cdot\frac{12}{13}\gtreqless 1,$$

or as $$r\lesseqgtr 4.$$

Thus the 4th and 5th terms are the greatest.

28. $$\frac{(r+1)\text{th term}}{r\text{th term}}=\frac{5r+3}{5r}\cdot\frac{22}{25},$$

and $$\frac{5r+3}{5r}\cdot\frac{22}{25}\gtrless 1,$$

as $$r\lessgtr 4\tfrac{2}{5}.$$

Thus after the 5th term each term is less than the preceding.

29. Since

$$128=125\left(1+\frac{3}{125}\right)=125(1+\cdot024),$$

$$\sqrt[3]{128}=\sqrt[3]{125}\,\sqrt[3]{1+\cdot024}=5(1+\cdot024)^{\frac{1}{3}}$$

$$=5\left\{1+\frac{1}{3}(\cdot024)-\frac{1.2}{3.6}(\cdot024)^2+\frac{1.2.5}{3.6.9}(\cdot024)^3-\dots\right\}$$

$$=5[1+\cdot008-\cdot000064+\cdot00000085-\dots]$$

$$=5\cdot0396842\dots,$$

$$33^{\frac{1}{5}}=32^{\frac{1}{5}}\left(1+\frac{1}{32}\right)^{\frac{1}{5}}$$

$$=2\left[1+\frac{1}{5}\cdot\frac{1}{32}-\frac{1.4}{5.10}\left(\frac{1}{32}\right)^2+\frac{1.4.9}{5.10.15}\left(\frac{1}{32}\right)^3-\dots\right]$$

$$=2[1+\cdot00625-\cdot000078]$$

$$=2\cdot01234\dots.$$

30. The different types of terms will clearly be the coefficient of x^4 in the continued product of the four expressions

$$1+ax+a^2x^2+a^3x^3+a^4x^4,$$
$$1+bx+b^2x^2+b^3x^3+b^4x^4,$$
$$1+cx+c^2x^2+c^3x^3+c^4x^4,$$
and
$$1+dx+d^2x^2+d^3x^3+d^4x^4.$$

The *number* of the terms will therefore be the coefficient of x^4 in

$$(1+x+x^2+x^3+x^4)^4,$$

i.e. the coefficient of x^4 in $(1-x^5)^4(1-x)^{-4}$,

i.e. $$(1-4x^5+\dots)\left(1+4x+\frac{4.5}{1.2}x^2+\frac{4.5.6}{1.2.3}x^3+\frac{4.5.6.7}{1.2.3.4}x^4+\dots\right).$$

Hence the required number of terms is 35.

The coefficient of the term *abcd* is ${}_4P_4=4.3.2.1.$

31. The term required is

$$\frac{\underline{|9}}{\underline{|3}\,\underline{|2}\,\underline{|4}}x^3(-y)^2(-z)^4.$$

$\therefore$ coefficient $$=\frac{\underline{|9}}{\underline{|3}\,\underline{|2}\,\underline{|4}}=1260.$$

32. $(1+x+x^2)^4=(1-x^3)^4(1-x)^{-4}$

$$=(1-4x^3+6x^6-\dots)\left(1+\frac{4}{1}x+\frac{4.5}{1.2}x^2+\frac{4.5.6}{1.2.3}x^3+\dots+\frac{4.5.6.7.8}{1.2.3.4.5}x^5+\dots\right).$$

Hence coefficient of $x^5=\frac{4.5.6.7.8}{1.2.3.4.5}-4\times\frac{4.5}{1.2}=16.$

Or thus: $(1+x+x^2)^4=1+4x(1+x)+6x^2(1+x)^2+4x^3(1+x)^3+x^4(1+x)^4$
$$=1+4x+\dots+x^5(4\times3+4)+\dots.$$

33. $$(1+x+x^5)^{10}=1+10x(1+x^4)+\frac{10.9}{1.2}x^2(1+x^4)^2+\dots$$
$$\dots+\frac{\underline{|10}}{\underline{|6}\,\underline{|4}}x^6(1+x^4)^6+\dots.$$

Hence coefficient of $x^6=\frac{10.9}{1.2}\times2+\frac{\underline{|10}}{\underline{|6}\,\underline{|4}}=300.$

34. $$(1-2x+x^2)^{-\frac{1}{3}}=(1-x)^{-\frac{2}{3}}.$$

Coefficient of $x^4=\frac{2.5.8.11}{3.6.9.12}.$

35. $(1+2x+2x^2)^4=1+4.2x(1+x)+6.4x^2(1+x)^2$
$$+4.8x^3(1+x)^3+16x^4(1+x)^4.$$

Hence coefficient of $x^4=6\times4+4\times8\times3+16=136.$

36. $$(1+x+x^3)^5=1+5x(1+x^2)+10x^2(1+x^2)^2$$
$$+10x^3(1+x^2)^3+5x^4(1+x^2)^4+x^5(1+x^2)^5.$$

Hence coefficient of $x^6=10+5\times 4=30$.

37. $$(1+x+x^2+x^3)^4=(1-x^4)^4(1-x)^{-4}$$
$$=(1-4x^4+\text{higher powers})\left(1+4x+\dots+\frac{4.5.6.7}{1.2.3.4}x^4+\dots\right).$$

Hence coefficient of $x^4=\dfrac{4.5.6.7}{1.2.3.4}-4=31$.

38. The general term

$$=\frac{\frac{3}{2}.\frac{5}{2}.\frac{7}{2}\dots\frac{2r+1}{2}}{1.2.3\dots r}2^{2r}.x^r=\frac{3.5.7\dots(2r+1).r!}{r!\,r!}2^rx^r$$
$$=\frac{3.5.7\dots(2r+1).2.4.6\dots2r}{r!\,r!}x^r=\frac{(2r+1)!}{r!\,r!}x^r.$$

39. The coefficient of x^n is negative and numerically equal to

$$\frac{1.3.5\dots(2n-1)}{2.4.6\dots.2n}2^{2n}=\frac{1.3.5\dots(2n-1)}{n!}2^n=\frac{1.3.5\dots(2n-1).2^n.n!}{n!\,n!}$$
$$=\frac{1.3.5\dots(2n-1).2.4.6\dots2n}{n!\,n!}=\frac{2n!}{n!\,n!}=\frac{2(2n-1)!}{n!\,(n-1)!}.$$

40. $$\frac{1}{n}.n+\frac{1}{n-1}.\frac{n(n-1)}{1.2}+\dots+\frac{1}{n-r+1}.\frac{n!}{r!\,(n-r)!}+\dots+1$$
$$=1+\frac{n}{1.2}+\dots+\frac{n!}{r!\,(n+1-r)!}+\dots+1$$
$$=\frac{1}{n+1}\left[1+\frac{n+1}{1}+\frac{(n+1)n}{1.2}+\dots+\frac{(n+1)!}{r!\,(n+1-r)!}+\frac{n+1}{1}+1\right]-\frac{2}{n+1}$$
$$=\frac{1}{n+1}(1+1)^{n+1}-\frac{2}{n+1}.$$

41. $$(1+2)^{n+1}=1+(n+1)2+\frac{(n+1)n}{1.2}2^2+\dots$$
$$+\frac{(n+1)!}{r!\,(n+1-r)!}2^r+\dots+(n+1)2^n+2^{n+1};$$
$$\therefore\ \frac{3^{n+1}-1}{n+1}=2C_0+2^2.\frac{C_1}{2}+\dots2^r.\frac{C_{r-1}}{r}+\dots+2^{n+1}\frac{C_n}{n+1}.$$

42. If $x<1$,

$$\sqrt{x^2+4}-\sqrt{x^2+1}=2\left(1+\frac{x^2}{4}\right)^{\frac{1}{2}}-(1+x^2)^{\frac{1}{2}}$$

$$=2\left[1+\frac{1}{2}\cdot\frac{x^2}{4}-\frac{1\,.\,1}{2\,.\,4}\,\frac{x^4}{16}+\dots\right]-\left[1+\frac{1}{2}\cdot x^2-\frac{1\,.\,1}{2\,.\,4}\,x^4+\dots\right]$$

$$=1-\frac{1}{4}x^2+\frac{7}{64}x^4-\dots.$$

Also, if $x>2$,

$$\sqrt{x^2+4}-\sqrt{x^2+1}=x\left(1+\frac{4}{x^2}\right)^{\frac{1}{2}}-x\left(1+\frac{1}{x^2}\right)^{\frac{1}{2}}$$

$$=x\left[1+\frac{1}{2}\cdot\frac{4}{x^2}-\frac{1\,.\,1}{2\,.\,4}\cdot\frac{16}{x^4}+\frac{1\,.\,1\,.\,3}{2\,.\,4\,.\,6}\cdot\frac{64}{x^6}-\dots\right]$$

$$-x\left[1+\frac{1}{2}\cdot\frac{1}{x^2}-\frac{1\,.\,1}{2\,.\,4}\cdot\frac{1}{x^4}+\frac{1\,.\,1\,.\,3}{2\,.\,4\,.\,6}\cdot\frac{1}{x^6}+\dots\right]$$

$$=\frac{3}{2}\frac{1}{x}-\frac{15}{8}\cdot\frac{1}{x^3}+\frac{63}{16}\cdot\frac{1}{x^5}-\dots.$$

43. $$(1-x)^{-1}(1-x^2)^{-p}(1-x^3)^{-q}(1-x^4)^{-r}$$

$$=[1+x+x^2+x^3+x^4+\dots]\left[1+px^2+\frac{1}{2}p(p+1)x^4+\dots\right]$$

$$[1+qx^3+\dots][1+rx^4+\dots]$$

$$=1+x+(p+1)x^2+(p+1+q)x^3+\left[p+1+\frac{1}{2}p(p+1)+q+r\right]x^4+\dots.$$

If this is equal to

$$1+x+2px^2+2qx^3+2rx^4+\dots,$$

we have $$p+1=2p,\ \text{and}\ \therefore\ p=1,$$

$$p+1+q=2q,\ \text{and}\ \therefore\ q=2,$$

$$p+1+\frac{1}{2}p(p+1)+q+r=2r,\ \text{and}\ \therefore\ r=5.$$

44. The coefficient is

$$\frac{1\,.\,3\,.\,5\dots(2r+1)}{2\,.\,4\,.\,6\dots2r}\,4^r=\frac{1\,.\,3\,.\,5\dots(2r+1)}{r\,!}\,2^r=\frac{1\,.\,3\,.\,5\dots(2r+1)\,.\,2^r\,.\,r\,!}{r\,!\,r\,!}$$

$$=\frac{1\,.\,3\,.\,5\dots(2r+1)\,.\,2\,.\,4\,.\,6\dots2r}{r\,!\,r\,!}=\frac{(2r)\,!}{r\,!\,r\,!}=2r\,.\,(2r-1)(2r-2)\dots(r+1)/r\,!.$$

45. $$r\,.\,\frac{C_r}{C_{r-1}}=r\,.\,\frac{n\,!}{r\,!\,(n-r)\,!}\times\frac{(r-1)\,!\,(n-r+1)\,!}{n\,!}=n-r+1;$$

$$\therefore\ \Sigma r\,.\,\frac{C_r}{C_{r-1}}=n+(n-1)+\dots+(n-r+1)+\dots+1$$

$$=\frac{1}{2}n(n+1).$$

46. If $x^2<1$,

$$(1-x)^{-\frac{3}{2}}=1+\frac{3}{2}x+\frac{3\,.\,5}{2\,.\,4}x^2+\frac{3\,.\,5\,.\,7}{2\,.\,4\,.\,6}x^3+\dots\text{ to infinity.}$$

Put $x=\frac{1}{2}$; then

$$S=\left(1-\frac{1}{2}\right)^{-\frac{3}{2}}=2^{\frac{3}{2}}=2\sqrt{2}.$$

47. If u_r be the rth term, three consecutive terms beginning at the rth are

$$u_r,\quad u_r\,.\,\frac{n-r+1}{r}x,\quad u_r\,.\,\frac{(n-r+1)(n-r)}{r(r+1)}x^2.$$

If these are in continued proportion

$$\frac{(n-r+1)(n-r)}{r(r+1)}=\left(\frac{n-r+1}{r}\right)^2;$$

$$\therefore\ r(n-r)=(r+1)(n-r+1)\ \text{ for }\ n-r+1\neq 0.$$

Hence $$n=-1.$$

48. $$(1+nx^n)\,[1+2x+3x^2+\ldots+(p+1)\,x^p+\ldots].$$

If $p<n$, coefficient of x^p is $p+1$.

If $p=n+r$, coefficient of x^{n+r} is $n(r+1)+(n+r+1)$.

49. If $x^2<1$,

$$(1-x)^{-\frac{1}{2}}=1+\frac{1}{2}x+\frac{1\,.\,3}{2\,.\,4}x^2+\frac{1\,.\,3\,.\,5}{2\,.\,4\,.\,6}x^3+\ldots \text{ to infinity.}$$

Hence $$1+\frac{1}{6}+\frac{1\,.\,3}{6\,.\,12}+\frac{1\,.\,3\,.\,5}{6\,.\,12\,.\,18}+\ldots \text{ to infinity}$$

$$=\left(1-\frac{1}{3}\right)^{-\frac{1}{2}}=\sqrt{\frac{3}{2}}.$$

50. $$(1-x)^2(1-2x)^{-2}=(1-2x+x^2)\,\{1+2\,.\,2x+3\,.\,2^2x^2+\ldots$$
$$+\overline{r+1}\,.\,2^rx^r+\ldots\};$$

$$\therefore\ \text{coefficient of } x^r=(r+1)\,2^r-2\,.\,r\,.\,2^{r-1}+\overline{r-1}\,.\,2^{r-2}$$
$$=2^{r-2}\{4r+4-4r+r-1\}$$
$$=2^{r-2}(r+3).$$

51. $$\left(\frac{1+x}{1-x}\right)^3=\frac{1}{2}(1+3x+3x^2+x^3)\,(1\,.\,2+2\,.\,3x+\ldots+\overline{r+1}\;\overline{r+2}\;x^r+\ldots);$$

$$\therefore\ \text{coefficient of } x^n=\frac{1}{2}\{\overline{n+1}\;\overline{n+2}+3n(n+1)+3(n-1)\,n+(n-2)(n-1)\}$$
$$=4n^2+2.$$

52. $$(1-x)^{-\frac{1}{3}}=1+\frac{1}{3}x+\frac{1\,.\,4}{3\,.\,6}x^2+\frac{1\,.\,4\,.\,7}{3\,.\,6\,.\,9}x^3+\ldots.$$

Hence $$\left(1-\frac{9}{10}\right)^{-\frac{1}{3}}=1+\frac{3}{10}+\frac{3\,.\,12}{10\,.\,20}+\frac{3\,.\,12\,.\,21}{10\,.\,20\,.\,30}+\ldots.$$

53. $$(1-x)^{-\frac{1}{2}}=1+\frac{1}{2}x+\frac{1\,.\,3}{2\,.\,4}x^2+\frac{1\,.\,3\,.\,5}{2\,.\,4\,.\,6}x^3+\dots;$$

$$\therefore\ 1+\frac{3}{8}+\frac{3\,.\,9}{8\,.\,16}+\frac{3\,.\,9\,.\,15}{8\,.\,16\,.\,24}+\dots=\left(1-\frac{3}{4}\right)^{-\frac{1}{2}}=2.$$

54. $$\sqrt{\frac{a+x}{a-x}}=\left(1+\frac{x}{a}\right)\left(1-\frac{x^2}{a^2}\right)^{-\frac{1}{2}}$$

$$=\left(1+\frac{x}{a}\right)\left\{1+\frac{1}{2}\frac{x^2}{a^2}+\dots+\frac{1\,.\,3\,.\,5\dots(2r-1)}{2\,.\,4\,.\,6\dots2r}\left(\frac{x}{a}\right)^{2r}+\dots\right\}.$$

The coefficients of $\left(\frac{x}{a}\right)^{2r-2}$ and $\left(\frac{x}{a}\right)^{2r-1}$ in the above product will clearly be equal.

LXXV. Pages 345, 347.

3. $$\log 46854=4\cdot6707467,$$
$$\log 46855=4\cdot6707559.$$

$$\log 46854\cdot6=4\cdot6707467+\frac{6}{10}\text{ of }\cdot0000092$$
$$=4\cdot6707467+\cdot0000055;$$
$$\therefore\ \log\cdot0468546=\bar{2}\cdot6707522.$$

4. $$\log 58961\cdot4=4\cdot7705648+\frac{4}{10}\text{ of }\cdot0000074$$
$$=4\cdot7705648+\cdot0000030;$$
$$\therefore\ \log\cdot00589614=\bar{3}\cdot7705678.$$

5. $$\log\sqrt[5]{29}=\frac{1}{5}\log 29=\frac{1}{5}\text{ of }1\cdot4623980=\cdot2924796.$$

$\log 1\cdot9612-\log 1\cdot9611=\cdot0000221$, $\log N-\log 1\cdot9611=\cdot0000020$.
Hence $N=1\cdot96111$.

6. $$\log\sqrt[7]{100}=\frac{1}{7}\log 100=\frac{1}{7}\text{ of }2=\cdot2857142.$$

Now $$\log 1\cdot9308-\log 1\cdot9307=\cdot0000225,$$
$$\log N-\log 1\cdot9307=\cdot0000006.$$

Hence $$N=1\cdot93070.$$

7. $$\log\sqrt[5]{15}=\frac{1}{5}\log 15=\frac{1}{5}\{\log 10+\log 3-\log 2\}$$

$$=\frac{1}{5}(1+\cdot4771213-\cdot3010300)=\cdot2352182.$$

Now $\log 1{\cdot}7188 - \log 1{\cdot}7187 = {\cdot}0000252,$

and $\log N - \log 1{\cdot}7187 = {\cdot}0000181.$

$$\therefore\ N = 1{\cdot}71877, \text{ since } \frac{181}{252} = {\cdot}7\ldots.$$

8. $$\log \sqrt[3]{{\cdot}034277} = \frac{1}{3}\log {\cdot}034277$$

$$= \frac{1}{3}(\bar{2}{\cdot}5350028) = \bar{1}{\cdot}5116676,$$

$\log {\cdot}32484 - \log {\cdot}32483 = {\cdot}0000134,\quad \log N - \log {\cdot}32483 = {\cdot}0000115.$

Hence $$N = {\cdot}324838, \text{ since } \frac{115}{134} = {\cdot}8\ldots$$

9. Amount $= (1{\cdot}05)^{100}.\quad \therefore\ \log A = 100 \times \log 1{\cdot}05 = 2{\cdot}11893.$
$\log 131{\cdot}51 - \log 131{\cdot}50 = {\cdot}0000330,\quad \log A - \log 131{\cdot}50 = {\cdot}0000042.$

Hence $$A = 131{\cdot}501, \text{ since } \frac{42}{330} = {\cdot}1\ldots.$$

Thus the amount is £131. 10*s*.

10. $A = P(1{\cdot}04)^{18}$. Hence $A > 2P$, if $\log(1{\cdot}04)^{18} > \log 2$; that is, if
$$18 \times {\cdot}0170333 > {\cdot}3010300.$$

11. $A = 500(1 + {\cdot}02)^{20}$. Now $\log(1{\cdot}02)^{20} = 20 \log 1{\cdot}02 = {\cdot}172004.$
$\log 1{\cdot}4860 - \log 1{\cdot}4859 = {\cdot}0000292,\quad \log N - \log 1{\cdot}4859 = {\cdot}0000144;$

$$\therefore\ N = 1{\cdot}48595, \text{ since } \frac{144}{292} = {\cdot}5.$$

Hence $$A = 500 \times £1{\cdot}48595 = £742{\cdot}975.$$

12. $1000 = P(1{\cdot}03)^{15};\ \therefore\ P = 1000(1{\cdot}03)^{-15}.$
Hence $$\log P = \log 1000 - 15 \log 1{\cdot}03 = 3 - 15 \times {\cdot}0128372$$
$$= 3 - {\cdot}1925580 = 2{\cdot}8074420.$$

Now $\log 641{\cdot}87 - \log 641{\cdot}86 = {\cdot}0000068,\quad \log P - \log 641{\cdot}86 = {\cdot}0000017.$

Hence $$P = 641{\cdot}862, \text{ since } \frac{17}{68} = {\cdot}2\ldots.$$

13. $$\frac{3 \log 1728}{1 + \frac{1}{2}\log {\cdot}36 + \frac{1}{3}\log 8} = \frac{9 \log 12}{1 + \log \frac{6}{10} + \log 2}$$

$$= \frac{9(\log 3 + 2 \log 2)}{1 + \log 3 + \log 2 - 1 + \log 2} = 9.$$

14. $$x \log 2 = \log 125 = 3 \log 5 = 3 - 3 \log 2.$$

Hence $$x = \frac{3 - {\cdot}90309}{{\cdot}30103} = 6{\cdot}9657\ldots.$$

15.
$$\log_e \frac{1+y}{1-y} = 2\left(y+\frac{1}{3}y^3+\frac{1}{5}y^5+\dots\right).$$

$$\therefore\ \log_e \frac{1+\frac{1}{2}}{1-\frac{1}{2}} = 2\left(\frac{1}{2}+\frac{1}{3}\cdot\frac{1}{2^3}+\frac{1}{5}\cdot\frac{1}{2^5}+\dots\right);$$

$$\therefore\ \log_e 3 = 1+\frac{1}{3}\cdot\frac{1}{2^2}+\frac{1}{5}\cdot\frac{1}{2^4}+\dots.$$

16.
$$\log\sqrt[5]{(\cdot00072\times2\cdot7)} = \frac{1}{5}(\log\cdot00072+\log27-1)$$

$$=\frac{1}{5}(2\log3+3\log2-5+3\log3-1)$$

$$=\frac{1}{5}(5\log3+3\log2-6)$$

$$=\frac{1}{5}(2\cdot3856065+\cdot90309-6)$$

$$=\frac{1}{5}(-5+2\cdot2886965)=\bar{1}\cdot4577393,$$

$$\log\left(\frac{1}{2000}\right)^{13}=13\log\frac{1}{2000}=13(-\log2-3)$$

$$=-13\times3\cdot30103=-42\cdot91339$$

$$=\overline{43}\cdot08661.$$

Hence there are 42 cyphers before the first significant figure.

17. $\log\cdot06+\log(\cdot6)^2-\log4-\log54+5=(\log2+\log3-2)$
$+2(\log2+\log3-1)-2\log2-(3\log3+\log2)+5=1.$

18.
$$\log\cdot003125=\log\frac{1}{320}=-5\log2-1.$$

$$\log\left(\frac{3}{4}\right)^{\frac{5}{7}}=\frac{5}{7}\log\frac{3}{4}=\frac{5}{7}(\cdot4771213-\cdot60206)$$

$$=-\frac{5}{7}\times\cdot1249387=-\cdot0892419$$

$$=\bar{1}\cdot910758$$

$$=\log\cdot81425.$$

19.
$$\log20+7\log\frac{15}{16}+5\log\frac{24}{25}+3\log\frac{80}{81}$$

$=\log2+1+7(\log3+1-5\log2)+5(\log3+5\log2-2)$
$+3(3\log2+1-4\log3)$
$=1.$

20.
$$3^{2x}-5\,.\,3^{2x}-6=0;$$

i.e.
$$(3^x-3)(3^x-2)=0.$$

If $3^x=3$, $\qquad x=1$.

If $3^x=2$,

$$x\log 3=\log 2\,;$$

$$\therefore\ x=\frac{\log 2}{\log 3}=\frac{\cdot 30103}{\cdot 4771213}=\cdot 6309\ldots.$$

21. $\log 385=\log 5+\log 7+\log 11=\cdot 6989700+\cdot 8450980$
$+1\cdot 0413927=2\cdot 5854607$,

$$\log\frac{175}{121}=2\log 5+\log 7-2\log 11=\cdot 1602526.$$

From $5^x.\,7^{x-1}=11^{x+2}$, we have

$$x\log 5+(x-1)\log 7=(x+2)\log 11\,;$$

$$\therefore\ x=\frac{\log 7+2\log 11}{\log 5+\log 7-\log 11}=\frac{2\cdot 9278834}{\cdot 5026753}=5\cdot 824\ldots.$$

22. $$S_n=4\frac{\left(\frac{3}{2}\right)^n-1}{\frac{3}{2}-1}=8\left\{\left(\frac{3}{2}\right)^n-1\right\}.$$

Hence $$8\left\{\left(\frac{3}{2}\right)^n-1\right\}>8000\,;\ \text{or}\ \left(\frac{3}{2}\right)^n>1001\,;$$

$$\therefore\ n\,(\log 3-\log 2)>3,$$

$$n>\frac{3}{\log 3-\log 2}>\frac{3}{\cdot 17609}>17.$$

Thus 18 terms must be taken.

23. $\log_e(1+x+x^2+\ldots+x^{m-1})=\log(1-x^m)-\log(1-x)$

$$=x+\frac{x^2}{2}+\ldots+\frac{x^r}{r}+\ldots-\left(x^m+\frac{x^{2m}}{2}+\frac{x^{3m}}{3}+\ldots\right).$$

Hence coefficient of x^n is $\frac{1}{n}$ when n is not a multiple of m, and coefficien of x^n is $\frac{1}{n}-\frac{m}{n}$ when n is a multiple of m.

24. If x be the required logarithm $12^x=180$.

Hence $$x\log_{10}12=\log_{10}180\,;$$

$$\therefore\ x=\frac{\log_{10}180}{\log_{10}12}=\frac{2\cdot 2552726}{1\cdot 0791813}=2\cdot 089\ldots.$$

25. In n years £1 will amount to $(1\cdot 03)^n$ at 3 per cent. per annun compound interest.

Hence $$(1\cdot 03)^n>2\,;$$

$$\therefore\ n\log 1\cdot 03>\log 2,$$

$$n>\frac{\cdot 30103}{\cdot 0128372}>23\cdot 4.$$

Hence $$n=24.$$

26.

$$x\log 2=\log 81=4\log 3;$$

$$\therefore\ x=\frac{4\log 3}{\log 2}=\frac{1{\cdot}9084852}{{\cdot}30103}=6{\cdot}3398\ldots.$$

27.

$$\text{Amount}=£100\times(1{\cdot}05)^{10},$$

$$\log A=\log 100+10\log 1{\cdot}05$$

$$=2{\cdot}211893;$$

$$\therefore\ \text{amount}=£162{\cdot}89=£162.\ 17s.\ 10d.$$

28.

$$x=\sqrt[3]{510},$$

$$\log x=\frac{1}{3}\log 510=\frac{1}{3}(\log 3+\log 17+1)$$

$$=\frac{1}{3}({\cdot}4771213+1{\cdot}2304489+1)$$

$$={\cdot}9025234.$$

Now $\log 7{\cdot}9895={\cdot}9025196$, difference 38,

$\log 7{\cdot}9896={\cdot}9025250$, difference 54,

and $\frac{38}{54}={\cdot}7.$

Hence $x=7{\cdot}98957.$

29.

$$\log 875=\log 7+3\log 5\ \dots\dots\dots\dots\ \text{(i)},$$

$$\log 2={\cdot}30103=\log 5-1\ \dots\dots\dots\dots\ \text{(ii)},$$

$$\log 105=\log 3+\log 7+\log 5\ \dots\dots\dots\dots\ \text{(iii)}.$$

From (ii) we have $\log 2$ and $\log 5$, and then (i) gives $\log 7$, and then (iii) gives $\log 3$.

Thus $\log 2$, $\log 3$, $\log 4\ (=2\log 2)$, $\log 5$, $\log 6\ (=\log 2+\log 3)$, $\log 7$, $\log 8\ (=3\log 2)$ and $\log 9\ (=2\log 3)$ are all found.

30.

$$\log 2+\log 3+\log 7=1{\cdot}6232493,$$

$$2\log 2+\log 3=1{\cdot}0791813,$$

$$2\log 7+2\log 3=2{\cdot}6444386;$$

$$\therefore\ 2\log 7+2\log 2+2\log 3=3{\cdot}2464986,$$

$$\log 7+2\log 2+2\log 3=1{\cdot}0791813+1{\cdot}3222193$$

$$=2{\cdot}4014006.$$

Hence $\log 7={\cdot}8450980.$

31. $10(\log 3-\log 2)+7(\log 5-2\log 3-\log 2)+4(\log 3+4\log 2-2\log 5)$

$$=-\log 2-\log 5=-\log 10=-1.$$

32.

$$\log N=7\log 1{\cdot}467799.$$

Now $\log 1{\cdot}4677={\cdot}1666373,$

$\log 1{\cdot}4678={\cdot}1666669$, difference 296;

$\therefore\ \log 1{\cdot}467799={\cdot}1666666,$

$\therefore\ \log N=1{\cdot}1666662,$

$\therefore\ N=14{\cdot}67798.$

33. $\log N = 1{\cdot}25 \log 277{\cdot}83$

$= 1{\cdot}25\,(3 \log 7 + 4 \log 3 - 2)$

$= 1{\cdot}25\,(2{\cdot}5352940 + 1{\cdot}9084852 - 2)$

$= 3{\cdot}0547240.$

Now $\log 1134{\cdot}2 = 3{\cdot}0546896,$

and $\log 1134{\cdot}3 = 3{\cdot}0547279,$ difference 383.

Hence $N = 1134{\cdot}29.$

34. $A = 200\,(1{\cdot}04)^{50}.$

Hence $\log A = \log 200 + 50 \log 1{\cdot}04$

$= 3{\cdot}15269695.$

Now $\log 1421{\cdot}3 = 3{\cdot}1526858,$

$\log 1421{\cdot}4 = 3{\cdot}1527163,$

and $\dfrac{111}{305} = {\cdot}3.$

Hence amount $= £1421{\cdot}33,$

and ∴ compound interest $= £1421{\cdot}33 - £200$

$= £1221.\ 6s.\ 8d.$

LXXVI. Pages 349, 350.

1. $x^2(x-1)-(x-1)=(x^2-1)(x-1)=$ &c.

2. $x^4(x^2-1)-(x^2-1)=(x^4-1)(x^2-1)=$ &c.

3. $(1-x)-x^4(1-x)=(1-x)(1-x^4)=$ &c.

4. $(1-x^2)-x^8(1-x^2)=(1-x^2)(1-x^8)=(1-x^2)(1-x^4)(1+x^4)=$ &c.

5. $(x-a)^2-(y+b)^2=$ &c.

6. $\left(x-\dfrac{3}{2}\right)^2-\left(y+\dfrac{1}{2}\right)^2=$ &c.

7. $x^2-2x(y+1)-8y^2+20y-8=x^2-2x(y+1)+(y+1)^2-9y^2+18y-9$

$=\{x-(y+1)\}^2-(3y-3)^2=(x+2y-4)(x-4y+2).$

8. $(a-c)^2-(b+f)^2=$ &c.

9. Cf. end of § 266.

10. x^n-a^n is divisible by $x-a$; ∴ (i) 7^n-1^n is divisible by $7-1$, and (ii) $35^{2n+1}-(-1)^{2n+1}$ is divisible by $35-(-1)$.

11. Given expression

$=\{3x^2-2x+1+(2x^2+3x-5)\}\times\{3x^2-2x+1-(2x^2+3x-5)\}.$

12. $(2a+3b)^3+(3a+2b)^3\div(2a+3b)+(3a+2b)$

$=(2a+3b)^2-(2a+3b)(3a+2b)+(3a+2b)^2=$ &c.

13. $(2a+4b-4c)^3+(a-b+7c)^3\div\dfrac{1}{3}\{(2a+4b-4c)+(a-b+7c)\}$

$=3(2a+4b-4c)^2-3(2a+4b-4c)(a-b+7c)+3(a-b+7c)^2=$ &c.

14. $1-x-x^5+x^6=(1-x)(1-x^5)=(1-x)(1-x)(1+x+x^2+x^3+x^4)$.

15. $1-x-x^n+x^{n+1}=(1-x)(1-x^n)=(1-x)(1-x)(1+x+\ldots+x^{n-1})$.

16. (i) $nx^{n+1}-(n+1)x^n+1=(x-1)nx^n-(x^n-1)$

$$=(x-1)\{nx^n-(x^{n-1}+x^{n-2}+\ldots+1)\}.$$

But since $(x-1)$, (x^2-1), (x^3-1), &c. are all divisible by $(x-1)$, so also is

$$x^{n-1}(x-1)+x^{n-2}(x^2-1)+x^{n-3}(x^3-1)+\ldots+(x^n-1),$$

that is $nx^n-(x^{n-1}+x^{n-2}+\ldots+1)$; $\therefore$ &c.

(ii) $x^n-nx+n-1=x^n-1-n(x-1)=(x-1)\{(x^{n-1}+x^{n-2}+\ldots+1)-n\}$.

As before, $(x-1)$ is a factor of $(x^{n-1}-1)+(x^{n-2}-1)+\ldots+(x-1)$; that is, of $(x^{n-1}+x^{n-2}+\ldots+1)-n$.

PAGE 357.

Ex. 3. Assume $$\frac{4x+7}{(x-3)(x-4)}=\frac{A}{x-3}+\frac{B}{x-4};$$

then $$4x+7\equiv A(x-4)+B(x-3).$$

In this identity put $x=4$; then $23=B$.

Put $x=3$; then $19=-A$.

Thus $$\frac{4x+7}{(x-3)(x-4)}=\frac{23}{x-4}-\frac{19}{x-3}.$$

Ex. 4. Assume $$\frac{x+1}{(x-2)(x-3)(x-4)}=\frac{A}{x-2}+\frac{B}{x-3}+\frac{C}{x-4}.$$

Then $x+1\equiv A(x-3)(x-4)+B(x-4)(x-2)+C(x-2)(x-3)$.

In this identity put $x=2$; then $3=2A$.

Put $x=3$; then $4=-B$.

Put $x=4$; then $5=2C$.

Hence $$\frac{x+1}{(x-2)(x-3)(x-4)}=\frac{3}{2(x-2)}-\frac{4}{x-3}+\frac{5}{2(x-4)}.$$

Ex. 5. $$\frac{x}{1-(a+b)x+abx^2}=\frac{1}{a-b}\left\{\frac{1}{1-ax}-\frac{1}{1-bx}\right\}$$

$$=\frac{1}{a-b}\{1+ax+a^2x^2+\ldots-(1+bx+b^2x^2+\ldots)\}.$$

$\therefore$ coefficient of x^r is $\frac{a^r-b^r}{a-b}$.

[a^2x^2 and b^2x^2 must both be <1.]

Ex. 6. $$\frac{x+5}{(x-1)(x+1)(x+2)}=\frac{1}{x-1}-\frac{2}{x+1}+\frac{1}{x+2}$$

$$=-\frac{1}{1-x}-\frac{2}{1+x}+\frac{1}{2}\cdot\frac{1}{1+\frac{x}{2}}$$

$$\begin{aligned}
&= - \quad (1 + x + x^2 + \ldots + x^{2n-1} + \ldots) \\
&\quad - 2 \; (1 - x + x^2 - \ldots - x^{2n-1} + \ldots) \\
&\quad + \frac{1}{2}\left(1 - \frac{x}{2} + \frac{x^2}{2^2} - \ldots - \frac{x^{2n-1}}{2^{2n-1}} + \ldots\right).
\end{aligned}$$

$\therefore$ coefficient of x^{2n-1} is $1 - \dfrac{1}{2^{2n}}$.

[In the above expansions x^2 must be < 1.]

Ex. 7.

$$\begin{aligned}
\frac{1}{6 - 7x + 2x^2} &= \frac{2}{3 - 2x} - \frac{1}{2 - x} \\
&= \frac{2}{3}\left(1 - \frac{2}{3}x\right)^{-1} - \frac{1}{2}\left(1 - \frac{x}{2}\right)^{-1} \\
&= \frac{2}{3}\left\{1 + \frac{2}{3}x + \ldots + \left(\frac{2}{3}x\right)^r + \ldots\right\} \\
&\quad - \frac{1}{2}\left\{1 + \frac{x}{2} \quad + \ldots + \left(\frac{x}{2}\right)^r + \ldots\right\}.
\end{aligned}$$

Hence coefficient of x^r in the required expansion $\left(\text{which requires that } \frac{2}{3}x \text{ should be numerically less than } 1\right)$ is $\left(\dfrac{2}{3}\right)^{r+1} - \left(\dfrac{1}{2}\right)^{r+1}$.

LXXVII. PAGES 366—369.

1. The expression $a^2(b-c) + b^2(c-a) + c^2(a-b)$ is of the third degree in a, b, and c, and $\therefore$ contains only three factors.

If $b = c$, the expression vanishes; $\therefore$ $(b-c)$ is a factor. Similarly $(c-a)$ and $(a-b)$ are factors.

$$\therefore \; a^2(b-c) + b^2(c-a) + c^2(a-b) = L(b-c)(c-a)(a-b),$$

where L is numerical.

Equating the coefficients of a^2b on each side, we find $L = -1$, &c.

Exactly the same argument holds for the other expression.

2. As in 1, $(b-c)$, $(c-a)$ and $(a-b)$ are factors.

But the expression is symmetrical and of the 5th degree. Thus we must have in addition a symmetrical factor of the second degree; *i.e.*

$$a^4(b-c) + b^4(c-a) + c^4(a-b)$$
$$= (b-c)(c-a)(a-b)\{L(a^2+b^2+c^2) + M(bc+ca+ab)\}.$$

Equating the coefficients of a^4b, we have $1 = -L$;

.... a^3b^2, ... $0 = L - M$;

whence, $L = -1$, $M = -1$; $\therefore$ &c.

3. As in 2, we have

$$(b-c)^5+(c-a)^5+(a-b)^5$$
$$=(b-c)(c-a)(a-b)\{L(a^2+b^2+c^2)+M(bc+ca+ab)\}.$$

Equating the coefficients of a^4b, we have $-5=-L$;

.... a^3b^2, ... $10=L-M$;

whence $L=5,\ M=-5$, &c.

4. Putting $a=0$, the expression vanishes; $\therefore$ a is a factor; and similarly, for b and c. But the expression is of the fourth degree and is symmetrical; thus the remaining factor is symmetrical and of the first degree, and $\therefore$ must be $(a+b+c)$.

Whence $(a+b+c)^4-(b+c)^4-$&c.$=Labc\,(a+b+c)$, where L is numerical.

Equating the coefficients of a^2bc, we have (by § 251),

$$\frac{\underline{|4}}{\underline{|2}\,\underline{|1}\,\underline{|1}}=L, \text{ \&c.}$$

5. As before $(b-c)$, $(c-a)$, and $(a-b)$ are factors; and the expression being of the 4th degree and symmetrical we must have a 4th factor $(a+b+c)$;

$$\therefore\ a(b-c)^3+b(c-a)^3+c(a-b)^3=L(b-c)(c-a)(a-b)(a+b+c).$$

Equating coefficients of ab^3, we have $1=L$, &c.

6. As in 2, we have $b^2c^2(b-c)+c^2a^2(c-a)+a^2b^2(a-b)$

$$=(b-c)(c-a)(a-b)\{L(a^2+b^2+c^2)+M(bc+ca+ab)\}.$$

Equating coefficients of a^4b, $0=-L$;

.... a^3b^2, $1=L-M$;

whence $L=0,\ M=-1$; $\therefore$ &c.

7. Exactly the same argument as in 6.

8. From 1, $a^2(b-c)+b^2(c-a)+c^2(a-b)=-(b-c)(c-a)(a-b)$;
in this equation write a^2 for a, b^2 for b, c^2 for c;

$$\therefore\ a^4(b^2-c^2)+b^4(c^2-a^2)+c^4(a^2-b^2)=-(b^2-c^2)(c^2-a^2)(a^2-b^2)=\text{\&c.}$$

9. As in 5, $bc(b^2-c^2)+ca(c^2-a^2)+ab(a^2-b^2)$

$$=L(b-c)(c-a)(a-b)(a+b+c).$$

Equating coefficients of a^3b, we have $1=-L$; $\therefore$ &c.

10. Putting $a=0$, the expression vanishes; $\therefore a$ is one factor. Similarly for b and c. But there can be only 3 factors,

$$\therefore\ a(b+c-a)^2+b(c+a-b)^2+\text{\&c.}=Labc.$$

Putting $a=b=c=1$, we have $4=L$; $\therefore$ &c.

11. As in 10, we obtain $a^2(b+c-a)+$&c.$=Labc$. Putting $a=b=c=1$, we have $2=L$, &c.

12. As in 10, we find a, b, and c are factors; a fourth symmetrical factor is required; thus

$$a\,(b+c)\,(b^2+c^2-a^2)+\&c.=Labc\,(a+b+c).$$

Equating coefficients of a^2bc, we have $0+1+1=L$; $\therefore$ &c.

13. (i) Putting $y=-z$, the expression becomes

$$x^{2n+1}-x^{2n+1}+z^{2n+1}-z^{2n+1}=0;\ \therefore\ (y+z) \text{ is a factor.}$$

Similarly $(z+x)$ and $(x+y)$ are factors.

(ii) When $n=1$, we have

$$(x+y+z)^3-x^3-y^3-z^3=L\,(y+z)\,(z+x)\,(x+y).$$

Equating coefficients of x^2y, we have (by § 251),

$$\frac{\underline{|3}}{\underline{|2}\,\underline{|1}}=L;\ \therefore\ L=3.$$

(iii) When $n=2$, we have $(x+y+z)^5-x^5-y^5-z^5$

$$=(y+z)\,(z+x)\,(x+y)\,\{L\,(x^2+y^2+z^2)+M\,(yz+zx+xy)\}.$$

Equating coefficients of x^4y, we have (by § 251), $\dfrac{\underline{|5}}{\underline{|4}\,\underline{|1}}=L$;

.... x^3y^2, ... $\dfrac{\underline{|5}}{\underline{|3}\,\underline{|2}}=L+M$;

whence $L=5$, $M=5$, &c.

14. $(x+y)^3-x^3-y^3=3x^2y+3xy^2=3xy\,(x+y)$.

$$(x+y)^5-x^5-y^5=5x^4y+10x^3y^2+10x^2y^3+5xy^4$$
$$=5xy\,(x^3+2x^2y+2xy^2+y^3)=5xy\,(x+y)\,(x^2+xy+y^2).$$

15. $$\frac{-a^3\,(b-c)-b^3\,(c-a)-c^3\,(a-b)}{(b-c)\,(c-a)\,(a-b)}=\frac{(b-c)\,(c-a)\,(a-b)\,(a+b+c)}{(b-c)\,(c-a)\,(a-b)}$$
$$=a+b+c.$$

16. $$\frac{-bc\,(b-c)-ca\,(c-a)-ab\,(a-b)}{(b-c)\,(c-a)\,(a-b)}=\frac{(b-c)(c-a)(a-b)}{(b-c)(c-a)(a-b)}=1.$$

17. This is the expression in 16, divided by abc.

18. $$\frac{-b^2c^2\,(b-c)-c^2a^2\,(c-a)-a^2b^2\,(a-b)}{a^2b^2c^2\,(b-c)\,(c-a)\,(a-b)}$$
$$=\frac{(b-c)\,(c-a)\,(a-b)\,(bc+ca+ab)}{a^2b^2c^2\,(b-c)\,(c-a)\,(a-b)}\ \text{[from 6]}=\&c.$$

19. $$\frac{-(b+c)\,(b-c)-(c+a)\,(c-a)-(a+b)\,(a-b)}{(b-c)\,(c-a)\,(a-b)}=0.$$

20. $$\frac{-bc(b+c)(b-c)-ca(c+a)(c-a)-ab(a+b)(a-b)}{abc(b-c)(c-a)(a-b)}$$
$$=\frac{(b-c)(c-a)(a-b)(a+b+c)}{abc(b-c)(c-a)(a-b)}=\&c.$$

21. The expression on the right
$$=\frac{-(x+b)(x+c)(b-c)-(x+c)(x+a)(c-a)-(x+a)(x+b)(a-b)}{(b-c)(c-a)(a-b)(x+a)(x+b)(x+c)}.$$
This numerator $= -x^2\{(b-c)+(c-a)+(a-b)\}$
$$-x\{(b^2-c^2)+(c^2-a^2)+(a^2-b^2)\}-bc(b-c)-ca(c-a)-ab(a-b)$$
$$=(b-c)(c-a)(a-b);\ \therefore\ \&c.$$

22. $$\frac{-a(x+b)(x+c)(b-c)-b(x+c)(x+a)(c-a)-c(x+a)(x+b)(a-b)}{(b-c)(c-a)(a-b)(x+a)(x+b)(x+c)}.$$
This numerator $= -x^2\{a(b-c)+b(c-a)+c(a-b)\}$
$$-x\{a(b^2-c^2)+b(c^2-a^2)+c(a^2-b^2)\}-abc\{(b-c)+(c-a)+(a-b)\}$$
$$=-x(b-c)(c-a)(a-b);\ \therefore\ \&c.$$

23. $$\frac{-a^2(x+b)(x+c)(b-c)-b^2(x+c)(x+a)(c-a)-c^2(x+a)(x+b)(a-b)}{(b-c)(c-a)(a-b)(x+a)(x+b)(x+c)}.$$
This numerator $= -x^2\{a^2(b-c)+b^2(c-a)+c^2(a-b)\}$
$$-x\{a^2(b^2-c^2)+b^2(c^2-a^2)+c^2(a^2-b^2)\}-abc\{a(b-c)+b(c-a)+c(a-b)\}$$
$$=x^2(b-c)(c-a)(a-b);\ \therefore\ \&c.$$

24. We find in the usual way that
$$(b-c)^3+(c-a)^3+(a-b)^3=L(b-c)(c-a)(a-b);$$
and comparing coefficients of a^2b we have $L=3$;
$$\therefore\ (b-c)^3+(c-a)^3+(a-b)^3=3(b-c)(c-a)(a-b).$$
Thence also,
$$(b^2-c^2)^3+(c^2-a^2)^3+(a^2-b^2)^3=3(b^2-c^2)(c^2-a^2)(a^2-b^2).$$

25. (i) $(a+b)(a-b)=a^2-b^2$; divide by $(a-b)$; $\therefore$ &c.

(ii) $(b-c)(c-a)(a-b)(a+b+c)=-a^3(b-c)-b^3(c-a)-c^3(a-b)$; [§ 276, Ex. 3]; divide by $(b-c)(c-a)(a-b)$; $\therefore$ &c.

(iii) We find that
$$a^4(b-c)(b-d)(c-d)-b^4(a-c)(a-d)(c-d)+c^4(a-b)(a-d)(b-d)$$
$$-d^4(a-b)(a-c)(b-c)$$
is divisible by $(a-b)$, $(a-c)$, $(a-d)$, $(b-c)$, $(b-d)$, and $(c-d)$.

Whence this expression must be equal to
$$(la+mb+nc+pd)(a-b)(a-c)(a-d)(b-c)(b-d)(c-d).$$

Equating the coefficients of a^4b^2c, b^4a^2c, c^4a^2b, and d^4a^2b successively, we find that $l=1=m=n=p$.

$$\therefore\ (a+b+c+d)(a-b)(a-c)\ldots\&c.=a^4(b-c)(b-d)(c-d)-\&c.$$

Now divide by $(a-b)(a-c)\ldots\&c.$

26. We find $a^3(b-c)(b-d)(c-d)-b^3(a-c)(a-d)(c-d)$

$$+c^3(a-b)(a-d)(b-d)-d^3(a-b)(a-c)(b-c)$$
$$=L(a-b)(a-c)(a-d)(b-c)(b-d)(c-d).$$

Equating coefficients of a^3b^2c, $L=1$; whence by dividing we obtain the required result.

27. We find $bcd(b-c)(b-d)(c-d)-cda(a-c)(a-d)(c-d)$

$$+dab(a-b)(a-d)(b-d)-abc(a-b)(a-c)(b-c)$$
$$=L(a-b)(a-c)(a-d)(b-c)(b-d)(c-d).$$

Whence, equating coefficients of a^3c^2d, $L=-1$; $\therefore$ &c.

28. Expression on the left

$$=\frac{-bc(b-c)(x-a)^2-ca(c-a)(x-b)^2-ab(a-b)(x-c)^2}{(b-c)(c-a)(a-b)}.$$

The numerator $=-x^2\{bc(b-c)+ca(c-a)+ab(a-b)\}$

$$+2abcx\{(b-c)+(c-a)+(a-b)\}-abc\{a(b-c)+b(c-a)+c(a-b)\}$$
$$=x^2(b-c)(c-a)(a-b);\ \therefore\ \&c.$$

29. Expression on the left

$$=\frac{1}{2}[\{(b+c)-(c+a)\}^2+\{(c+a)-(a+b)\}^2+\{(a+b)-(b+c)\}^2]$$
$$=\frac{1}{2}[(b-a)^2+(c-b)^2+(a-c)^2]=\&c.$$

30. $(b+c)^3+\&c.$ [by § 277]

$$=\{(b+c)+(c+a)+(a+b)\}\{(b+c)^2+(c+a)^2+(a+b)^2-(c+a)(a+b)$$
$$-(a+b)(b+c)-(b+c)(c+a)\}$$
$$=\{2(a+b+c)\}\{a^2+b^2+c^2-bc-ca-ab\}\ [\text{LXXVII. }29]$$
$$=\&c.$$

31. Applying the formula

$$\alpha^3+\beta^3+\gamma^3-3\alpha\beta\gamma=\frac{1}{2}(\alpha+\beta+\gamma)\{(\beta-\gamma)^2+(\gamma-\alpha)^2+(\alpha-\beta)^2\}$$

the expression on the left becomes

$$\frac{1}{2}(a+b+c)\{(2c-2b)^2+(2a-2c)^2+(2b-2a)^2\}$$
$$=4(a+b+c)[(a^2+b^2+c^2-bc-ca-ab)]=\&c.$$

32. The expression on the left is the product of two factors of which one is $(x^2+2yz)+(y^2+2zx)+(z^2+2xy)$, that is $(x+y+z)^2$; and the other is

$$(x^2+2yz)^2+(y^2+2zx)^2+(z^2+2xy)^2-(y^2+2zx)(z^2+2xy)$$
$$-(z^2+2xy)(x^2+2yz)-(x^2+2yz)(y^2+2zx)$$
$$=x^4+y^4+z^4-2x^3(y+z)-2y^3(z+x)-2z^3(x+y)+3y^2z^2+3z^2x^2+3x^2y^2$$
$$=(x^2+y^2+z^2-yz-zx-xy)^2.$$

Hence the expression on the left

$$=(x+y+z)^2(x^2+y^2+z^2-yz-zx-xy)^2=(x^3+y^3+z^3-3xyz)^2.$$

33. If $x^2+y^2+z^2=yz+zx+xy$; then

$2(x^2+y^2+z^2)-2(yz+zx+xy)=0$, *i.e.* $(y-z)^2+(z-x)^2+(x-y)^2=0$, which requires that $x=y=z$.

34. (i) If $2(a^2+b^2)=(a+b)^2$; then $a^2-2ab+b^2=0$, *i.e.* $(a-b)^2=0$.

(ii) If $3(a^2+b^2+c^2)=(a+b+c)^2$; then

$2a^2+2b^2+2c^2-2ab-2ac-2bc=0$, *i.e.* $(a-b)^2+(a-c)^2+(b-c)^2=0$, whence $a=b=c$.

(iii) and (iv) If $n(a^2+b^2+c^2+\dots)=(a+b+c+\dots)^2$; then

$$(n-1)(a^2+b^2+\dots)-2(ab+ac+\dots)=0;$$

$\therefore$ $(a-b)^2+(a-c)^2+\dots=0$, *every letter being taken with each of the* $n-1$ *other letters.* Hence $a=b=c=$&c.

35.

$$x+\frac{1}{x}=2+\left(\sqrt{x}-\frac{1}{\sqrt{x}}\right)^2=2+\text{square of a real quantity},$$
$$x+\frac{4}{x}=4+\left(\sqrt{x}-\frac{2}{\sqrt{x}}\right)^2=4+\dots\dots\dots\dots$$
$$x+\frac{9}{x}=6+\left(\sqrt{x}-\frac{3}{\sqrt{x}}\right)^2=6+\dots\dots\dots\dots$$

36. Put $\dfrac{x^2-3x+4}{x^2+3x+4}=\lambda$; then $x^2(1-\lambda)-3x(1+\lambda)+4(1-\lambda)=0$.

Now, since x is real, $\{3(1+\lambda)\}^2-16(1-\lambda)^2>0$, [Art. 147];

$$i.e.\ -(7\lambda^2+50\lambda-7)>0,\ i.e.\ -(7\lambda-1)(\lambda-7)>0.$$

Hence $7\lambda-1$ and $\lambda-7$ must be of different signs, and $\therefore$ λ must lie between $\frac{1}{7}$ and 7.

37. For, if they are not all equal, we can always, without altering the sum, make the continued product greater; *e.g.* take a and b, any two which are not equal, and replace them by $\dfrac{a+b}{2}$ and $\dfrac{a+b}{2}$.

Then the product of the two new quantities is

$$\left(\frac{a+b}{2}\right)^2 = ab + \left(\frac{a-b}{2}\right)^2 > ab;$$

thus we have made the continued product greater.

38. For, if they are not all equal, we can always, without altering the continued product, make the sum less; *e.g.* take a and b, any two which are not equal, and replace them by $\sqrt{ab}$ and $\sqrt{ab}$.

Then the sum of the two new quantities is $2\sqrt{ab}$, and since

$$(\sqrt{a}-\sqrt{b})^2 > 0; \;\; \therefore \; a+b > 2\sqrt{ab}.$$

Thus we have made the sum less.

39. Put $\dfrac{yz-x^2}{y+z} = \dfrac{zx-y^2}{z+x} = \lambda$.

Then $$yz - x^2 = \lambda(y+z), \;\; zx - y^2 = \lambda(z+x);$$

$$\therefore \; \lambda(y-x) = yz - zx + y^2 - x^2 = (y-x)(x+y+z).$$

Hence, *provided y is not equal to x*, $\lambda = x+y+z$ .... I.

Hence $$yz - x^2 = (y+z)(x+y+z),$$

whence $$x^2+y^2+z^2+yz+zx+xy=0 \text{ .... II.}$$

Then, from II. $$xy - z^2 = (x+y)(x+y+z),$$

whence $$(xy-z^2)/(x+y) = x+y+z = \lambda.$$

40. For $\{(a-b)+(c-d)\}(ad-bc) = \{(a-b)-(c-d)\}(ac-bd)$,

$$\therefore \; (a-b)\{(ad-bc)-(ac-bd)\} + (c-d)\{(ad-bc)+(ac-bd)\} = 0,$$

$$i.e. \; -(a-b)(a+b)(c-d) + (c-d)(a-b)(c+d) = 0;$$

$$\therefore \; (a-b)(c-d)(c+d-a-b) = 0; \;\; \therefore \; \&c.$$

41. We may write these equations as follows,

$$a^2(x+y+z-1) - 2a(\alpha x+\beta y+\gamma z-\delta) + (\alpha^2 x+\beta^2 y+\gamma^2 z-\delta^2) = 0, \text{ I}$$

$$b^2(x+y+z-1) - 2b(\alpha x+\beta y+\gamma z-\delta) + (\alpha^2 x+\beta^2 y+\gamma^2 z-\delta^2) = 0, \text{ II}$$

$$c^2(x+y+z-1) - 2c(\alpha x+\beta y+\gamma z-\delta) + (\alpha^2 x+\beta^2 y+\gamma^2 z-\delta^2) = 0. \text{ III}$$

$$(\text{I}-\text{II}) \div (a-b); \;\; \therefore \; (a+b)(x+y+z-1) - 2(\alpha x+\beta y+\gamma z-\delta) = 0; \text{ IV}$$

$$(\text{II}-\text{III}) \div (b-c); \;\; \therefore \; (b+c)(x+y+z-1) - 2(\alpha x+\beta y+\gamma z-\delta) = 0; \text{ V}$$

$$(\text{IV}-\text{V}) \div (a-c); \;\; \therefore \; x+y+z-1 = 0;$$

∴ from IV, $$\alpha x+\beta y+\gamma z-\delta = 0;$$

and ∴ from I, $$\alpha^2 x+\beta^2 y+\gamma^2 z-\delta^2 = 0.$$

Thence

$$d^2(x+y+z-1) - 2d(\alpha x+\beta y+\gamma z-\delta) + (\alpha^2 x+\beta^2 y+\gamma^2 z-\delta^2) = 0; \;\; \therefore \; \&c.$$

[See also *Treatise on Algebra*, Art. 127, Ex. 4.]

42. $\dfrac{a}{x+a} - \dfrac{c}{x+c} = \dfrac{d}{x+d} - \dfrac{b}{x+b}$; $\therefore\ \dfrac{x(a-c)}{(x+a)(x+c)} = \dfrac{x(d-b)}{(x+b)(x+d)}$;

whence $x\{x^2(a+b-c-d)+2x(ab-cd)+(abd-bcd+abc-acd)\}=0$, I.

Now one root of this equation is $x=0$; the other two roots will be equal (§ 145) if

$$4(ab-cd)^2-4(a+b-c-d)(abd-bcd+abc-acd)=0.$$

By the method of § 276, we find that this is equivalent to

$$4(a-c)(a-d)(b-c)(b-d)=0;$$
$$\therefore\ a=c, \text{ or } a=d, \text{ or } b=c \text{ or } b=d.$$

(i) If $a=c$, I becomes $x\{(b-d)(x^2+2ax+a^2)\}=0$;

If $a=d$, I becomes $x\{(b-c)(x^2+2ax+a^2)\}=0$;

and in each case the roots are 0, $-a$, $-a$.

(ii) Similarly, if $b=c$, or d, the roots are 0, $-b$, $-b$.

(iii) Two roots will be zero, if $abd-bcd+abc-acd=0$; *i.e.* if

$$\frac{1}{a}+\frac{1}{b}=\frac{1}{c}+\frac{1}{d}.$$

The third root will then be $-2\dfrac{ab-cd}{a+b-c-d}$.

But $a+b-c-d=cd\left(\dfrac{a+b}{cd}-\dfrac{1}{d}-\dfrac{1}{c}\right)=cd\left(\dfrac{a+b}{cd}-\dfrac{1}{a}-\dfrac{1}{b}\right)$

$=\dfrac{(a+b)(ab-cd)}{ab}$. $\therefore$ the third root is $-2\dfrac{ab}{a+b}$.

43. For we have,

$$x^2(y^2+2y'^2-2yy')-2xx'(y^2+y'^2-yy')+x'^2(2y^2+y'^2-2yy')=0;$$
$$\therefore\ (x^2-2xx'+x'^2)(y^2-2yy'+y'^2)+x^2y'^2-2xx'yy'+x'^2y^2=0,$$
$$i.e.\ (x-x')^2(y-y')^2+(xy'-x'y)^2=0.$$

Hence $\quad (x-x')(y-y')=0$, and $xy'-x'y=0$.

From the first relation $x=x'$, or $y=y'$; and either of these combined with the second relation shews that both $x=x'$, and $y=y'$.

44. We have $(a^2+3ad)(a^2+3ad+2d^2)+d^4$

$$=(a^2+3ad)^2+2d^2(a^2+3ad)+d^4=(a^2+3ad+d^2)^2.$$

45. (i) $\sqrt[3]{(x^3-24x^2y+192xy^2-512y^3)}=(x-8y)$. Cf. § 280.

(ii) Cf. § 282.

$$\begin{array}{r|l} & 1+3x+6x^2+7x^3+6x^4+3x^5+x^6 \\ 1^3= & 1 \\ (1+x)^3= & \overline{1+3x+3x^2+\ \ x^3} \\ (1+x+x^2)^3= & \overline{1+3x+6x^2+7x^3+6x^4+3x^5+x^6} \end{array}$$

(iii)
$$\begin{array}{rl} & x^6-9x^5+33x^4-63x^3+66x^2-36x+8 \\ (x^2)^3= & x^6 \\ \hline (x^2-3x)^3= & x^6-9x^5+27x^4-27x^3 \\ \hline (x^2-3x+2)^3= & x^6-9x^5+33x^4-63x^3+66x^2-36x+8 \end{array}$$

(iv)
$$\begin{array}{rl} & 8x^6-36x^5+102x^4-171x^3+204x^2-144x+64 \\ (2x^2)^3= & 8x^6 \\ \hline (2x^2-3x)^3= & 8x^6-36x^5+\ \ 54x^4-\ \ 27x^3 \\ \hline (2x^2-3x+4)^3= & 8x^6-36x^5+102x^4-171x^3+204x^2-144x+64 \end{array}$$

46. Expression on the left
$=(x+y+z-x)\,(x+y+z-y)\,(x+y+z-z)+xyz$
$=(x+y+z)^3-(x+y+z)^2\,(x+y+z)+(x+y+z)\,(yz+zx+xy)-xyz+xyz$
$=(x+y+z)\,(yz+zx+xy)=$ &c.

47. Since $(a^2+b^2)\,x^2-2b\,(a+c)\,x+b^2+c^2=0$, $(ax-b)^2+(bx-c)^2=0$;
$\therefore\ ax-b=0$ and also $bx-c=0$, &c.

48. Assume it true that
$$(1+x)\,(1+x^2)\,(1+x^{2^2})\ldots(1-x^{2^{n-1}})=\frac{1-x^{2^n}}{1-x}.$$
Then $(1+x)\,(1+x^2)\ldots(1+x^{2^{n-1}})\,(1+x^{2^n})=\dfrac{(1-x^{2^n})\,(1+x^{2^n})}{1-x}=\dfrac{1-x^{2^{n+1}}}{1-x}$.

Thus, if the theorem holds for any value of n, it will hold for the next greater value; $\therefore$ &c.

49. $as+bc=(a+b)\,(a+c)$, $bs+ca=(b+c)\,(b+a)$, $cs+ab=(c+a)\,(c+b)$;
$\therefore$ &c.

50. $(a^3+b^3+c^3+d^3)^2=\{a^3+b^3+c^3-(a+b+c)^3\}^2$
$=\{-3\,(b+c)\,(c+a)\,(a+b)\}^2=9\,(b+c)^2\,(c+a)^2\,(a+b)^2$.

Also $\{bcd+cda+dab+abc\}^2=\{abc-(bc+ca+ab)\,(a+b+c)\}^2$
$=\{-(b+c)\,(c+a)\,(a+b)\}^2$ [see 46, above].

Again, $bc-ad=bc+a\,(a+b+c)=(a+b)\,(a+c)$;
$\therefore\ (bc-ad)\,(ca-bd)\,(ab-cd)=(b+c)^2\,(c+a)^2\,(a+b)^2$.

LXXVIII. Page 368.

1.
$$\begin{array}{r} 6\,\underline{)\,2156} \\ 6\,\underline{)\,359\ldots2} \\ 6\,\underline{)\,59\ldots5} \\ 6\,\underline{)\,9\ldots5} \\ 1\ldots3 \end{array}
\qquad
\begin{array}{r} 6\,\underline{)\,7213} \\ 6\,\underline{)\,1202\ldots1} \\ 6\,\underline{)\,200\ldots2} \\ 6\,\underline{)\,33\ldots2} \\ 5\ldots3 \end{array}
\qquad
\begin{array}{r} 6\,\underline{)\,192457} \\ 6\,\underline{)\,32076\ldots1} \\ 6\,\underline{)\,5346\ldots0} \\ 6\,\underline{)\,891\ldots0} \\ 6\,\underline{)\,148\ldots3} \\ 6\,\underline{)\,24\ldots4} \\ 4\ldots0 \end{array}$$

Thus the numbers are 13552, 53221 and 4043001.

2. [See Art. 286, Ex. 2.]

$$2135 \text{ in scale } 7=\{(2\times7+1)\times7+3\}\times7+5=761,$$
$$4210 \text{ in scale } 7=\{(4\times7+2)\,7+1\}\times7=1477.$$

3. [See Art. 286, Ex. 6.]

$$\begin{array}{r} 12\,)\,\underline{3152} \\ 12\,)\,\underline{233}\ldots e \\ 12\,)\,\underline{17}\ldots 0 \\ 1\ldots 4 \end{array}\qquad\qquad \begin{array}{r} 12\,)\,\underline{23678} \\ 12\,)\,\underline{1728}\ldots 2 \\ 12\,)\,\underline{132}\ldots 2 \\ 12\,)\,\underline{10}\ldots 2 \\ 0\ldots 9 \end{array}$$

Thus the numbers are 140e and 9222.

4.

$$\begin{array}{r} 5\,)\,\underline{23} \\ 4\ldots 3 \end{array}\qquad\qquad \begin{array}{r} \cdot42 \\ \underline{5} \\ 2\cdot10 \\ \underline{5} \\ 0\cdot50 \\ \underline{5} \\ 2\cdot50 \end{array}$$

Hence 23·42 in scale 10 = $43\cdot20\dot{2}$ in scale 5.

$$\begin{array}{r} 5\,)\,\underline{123} \\ 5\,)\,\underline{24}\ldots 3 \\ 4\ldots 4 \end{array}\qquad\qquad \begin{array}{r} \cdot45 \\ \underline{5} \\ 2\cdot25 \\ \underline{5} \\ 1\cdot25 \end{array}$$

Hence 123·45 in scale 10 = $443\cdot2\dot{1}$ in scale 5.

5. Work in the scale of 6 throughout.

$F_1=2\cdot3\dot{1}$; $\therefore F_1\times6=23\cdot\dot{1}$, and $F_1\times6^2=231\cdot\dot{1}$.

$$\therefore F_1=\frac{231-23}{6^2-6}=\frac{204}{50}.\quad \text{Similarly } F_2=\frac{125-12}{50}=\frac{113}{50}.$$

Then $$\frac{204}{50}\times\frac{113}{50}=\frac{23500}{4100}=\frac{235}{41}=3\cdot\dot{4}.$$

6. Let the radix of the scale be r, then $121=r^2+2r+1=(r+1)^2$.

$$12321=r^4+2r^3+3r^2+2r+1=(r^2+r+1)^2,$$
$$1234321=r^6+2r^5+3r^4+4r^3+3r^2+2r+1=(r^3+r^2+r+1)^2.$$

7. By § 290, if the number is divisible by 11, $(2+5+1+b)-(1+a+4+3)$, *i.e.* $(b-a)$, must be zero or some multiple of 11. But a and b are both positive and less than 10; hence this condition requires that $b=a$.

By § 289, if the number is divisible by 9, $2+1+5+a+1+4+b+3$, that is $16+a+b$, or $16+2a$, must be divisible by 9; which requires that $a=1$.

8. As in 7, since the number is divisible by 11 we have,

$$14+b-a=0, \text{ or } 11, \text{ or } 22, \text{ \&c.}$$

$$\textit{i.e.} \quad b-a=-14, \text{ or } -3, \text{ or } 8, \text{ \&c.}$$

But since a and b are both less than 10, it is only possible that

$$b-a=-3, \text{ or } 8.$$

Again, since the number is divisible by 9, $30+a+b$ is a multiple of 9; $\therefore a+b=6$, or 15 (the other values being impossible).

The only integral solution is obtained from $b-a=-3$ and $a+b=15$, giving $a=9$ and $b=6$.

9. $314=6+2r+6r^2$; whence $r=7 \left(\text{or } -\frac{22}{3}\right)$.

10. If x and y are the digits; then $5y+x=2(5x+y)$; whence $y=3x$. and both being less than 5, the digits are 1 and 3.

11. We have $8y+x=2(8x+y)$; $\therefore 2y=5x$; $\therefore$ digits must be 2 and 5.

12. We have $7y+x=3(7x+y)$; $\therefore y=5x$; $\therefore$ digits must be 1 and 5.

MISCELLANEOUS EXAMPLES. VII. Pages 376—392.

1. The factors of the continued product can be taken in any order. Hence the product

$$=\{(1-x)(1+x)(1+x^2)(1+x^4)\}^2=(1-x^8)^2=1-2x^8+x^{16}.$$

2. (i) $4x^2-4x-15=(2x-5)(2x+3)$.

(ii) $(b-c)x^2+(c-a)x+a-b=\{(b-c)x-(a-b)\}(x-1)$.

(iii) The given expression is

$$\begin{aligned}&[xyz-a\Sigma yz+a^2\Sigma x-a^3](b-c)\\&+[xyz-b\Sigma yz+b^2\Sigma x-b^3](c-a)\\&+[xyz-c\Sigma yz+c^2\Sigma x-c^3](a-b)\\&=xyz(b-c+c-a+a-b)-\Sigma yz\{a(b-c)+b(c-a)+c(a-b)\}\\&\qquad+\Sigma x\{a^2(b-c)+b^2(c-a)+c^2(a-b)\}-\Sigma a^3(b-c)\\&=-\Sigma x\,.\,(b-c)(c-a)(a-b)+(a+b+c)(b-c)(c-a)(a-b)\end{aligned}$$

[See Ex. 3, p. 359.]

$$=(a+b+c-x-y-z)(b-c)(c-a)(a-b).$$

3. (i) $\dfrac{21}{x}-\dfrac{49}{y}=28$ and $-\dfrac{21}{x}+\dfrac{12}{y}=9$;

$$\therefore\ -\frac{37}{y}=37 \text{ and } \therefore\ y=-1.$$

Then $$\frac{3}{x}+7=4, \text{ and } \therefore\ x=-1.$$

(ii) We have $\dfrac{(a+c)(x+a)-a(x+a+c)}{(x+a+c)(x+a)}=\dfrac{b(x+b-c)-(b-c)(x+b)}{(x+b)(x+b-c)}$;

$$\therefore\ \frac{cx}{(x+a+c)(x+a)}=\frac{cx}{(x+b)(x+b-c)}.$$

Hence $x=0$,

or $$(x+a)(x+a+c)=(x+b)(x+b-c),$$

whence $$x=\frac{1}{2}(a+b).$$

(iii) $$(x^2-5x)^2+4(x^2-5x)+4=x^2-5x+22;$$

$$\therefore\ (x^2-5x+6)(x^2-5x-3)=0,$$

i.e. $$(x-2)(x-3)\left(x-\frac{5}{2}+\frac{1}{2}\sqrt{37}\right)\left(x-\frac{5}{2}-\frac{1}{2}\sqrt{37}\right)=0.$$

5. Each of the equal fractions $=\lambda$

$$\begin{aligned}&=\frac{ax-by-cz}{px(px-qy-rz)-qy(qy-rz-px)-rz(rz-px-qy)}\\&=\frac{ax-by-cz}{p^2x^2-(qy-rz)^2}=\frac{ax-by-cz}{(px-qy+rz)(px+qy-rz)}.\end{aligned}$$

Thus $$ax-by-cz=\lambda\,(px-qy+rz)\,(px+qy-rz)$$

and $$\frac{a}{p}=-\lambda\,(-px+qy+rz).$$

$$\therefore\ \frac{a\,(ax-by-cz)}{p}=-\lambda^2\,(-px+qy+rz)\,(px-qy+rz)\,(px+qy-rz).$$

Hence $a\,(ax-by-cz)/p=b\,(-ax+by-cz)/q=c\,(-ax-by+cz)/r.$

6. Put $$(x^2+1)/(x^2-x+1)=y;$$

then $$x^2\,(1-y)+xy+1-y=0.$$

The roots of this quadratic equation are real, and

$$\therefore\ y^2-4\,(1-y)^2 \not< 0,$$

i.e. $$(2-y)\,(3y-2)\not< 0.$$

Hence $$y \not> 2 \text{ and } y \not< \frac{2}{3}.$$

7. We have to find the sum of n terms of

$$(2+3)+(5+6)+(8+9)+\ldots,$$

i.e. of $$5+11+17+\ldots.$$

$$\therefore\ S=\frac{n}{2}\{10+(n-1)\,6\}=3n^2+2n.$$

8. First the whole number 231 will be changed into 133, as appears from

$$\begin{array}{r|l} 8 & 233 \\ \hline 8 & 15\ldots3 \\ \hline & \ \ 1\ldots3 \end{array}$$

And to change $\cdot 4\dot{3}\dot{1}$ from the scale of 6 to the scale of 8, we proceed as in Art. 288.

$$\begin{array}{r} \cdot 4\dot{3}\dot{1} \\ \hline 6\cdot\dot{0}\dot{2} \\ \hline 0\cdot\dot{2}\dot{4} \\ \hline 3\cdot\dot{3}\dot{5} \\ \hline 5\cdot\dot{1}\dot{3} \\ \hline 2\cdot\dot{0}\dot{2} \end{array}$$

Thus $\cdot 4\dot{3}\dot{1}$ is changed into $\cdot 6\dot{0}35\dot{2}$.

9. $$\frac{1}{6-5x+x^2}=\frac{1}{(3-x)\,(2-x)}=\frac{1}{2-x}-\frac{1}{3-x}=\frac{1}{2}\left(1-\frac{x}{2}\right)^{-1}-\frac{1}{3}\left(1-\frac{x}{3}\right)^{-1}.$$

$\therefore$ coefficient of x^n is $\dfrac{1}{2^{n+1}}-\dfrac{1}{3^{n+1}}$.

10. In the formula

$$\log_e\frac{m}{n}=2\left\{\frac{m-n}{m+n}+\frac{1}{3}\left(\frac{m-n}{m+n}\right)^3+\ldots\right\},$$

put $m=x^2-1$ and $n=x^2$; then

$$\log_e \frac{x^2-1}{x^2} = -2 \left\{\frac{1}{2x^2-1} + \frac{1}{3} \; \frac{1}{(2x^2-1)^3} + \dots\right\}.$$

Hence $\log_e (x+1) + \log_e (x-1) = 2 \log_e x - 2 \left\{\frac{1}{2x^2-1} + \frac{1}{3} \cdot \frac{1}{(2x^2-1)^3} + \dots\right\}.$

11. If $a+b+c+d=0$,

$$bc-ad=bc+a\ (a+b+c)=(a+b)\ (a+c).$$

Similarly $\qquad ca-bd=(b+c)\ (b+a)$,

and $\qquad ab-cd=(c+a)\ (c+b)$.

Hence $\qquad (bc-ad)\ (ca-bd)\ (ab-cd)=(b+c)^2\ (c+a)^2\ (a+b)^2$.

12.
$$S=\frac{a+x}{x\ (x-y)\ (x-z)}+\frac{a+y}{y\ (y-z)\ (y-x)}+\frac{a+z}{z\ (z-x)\ (z-y)}.$$

$\therefore\ S \times xyz\ (y-z)\ (z-x)\ (x-y)$

$$=yz\ (x+a)\ (z-y)+zx\ (y+a)\ (x-z)+xy\ (z+a)\ (y-x)$$
$$=xyz\ (z-y+x-z+y-x)+a\Sigma yz\ (z-y)$$
$$=a\ (y-z)\ (z-x)\ (x-y).$$

Hence $\qquad S=\dfrac{a}{xyz}$.

13. (i) $\qquad (b-a)^2 x-(b-a)\ (c-b)\ y=(b-a)\ (c-a)$,

and $\qquad (a-c)\ (c-b)\ x-(b-a)\ (c-b)\ y=(b-c)\ (c-b)$.

$$\therefore\ x=\frac{(b-a)\ (c-a)+(b-c)^2}{(c-a)\ (c-b)+(b-a)^2}=\frac{a^2+b^2+c^2-bc-ca-ab}{a^2+b^2+c^2-bc-ca-ab}=1.$$

Hence $\qquad b-a-(c-b)\ y=c-a$;

$$\therefore\ y=-1.$$

(ii) $\qquad x^2+(1-a^2)\ x-a^2=0$,

i.e. $\qquad (x-a^2)\ (x+1)=0$, &c.

(iii) In the second equation substitute $3-y$ for x; then

$$(3-y)^3+y^3=9\ (3-2y),$$

whence $\qquad 9y^2-9y=0$;

$$\therefore\ y=0 \text{ or } y=1.$$

If $y=0$, $\qquad x=3$;

and if $y=1$, $\qquad x=2$.

14. We have $\qquad \dfrac{7\sqrt{5}+5\sqrt{3}}{\sqrt{5}-\sqrt{3}}+\dfrac{7\sqrt{5}-5\sqrt{3}}{\sqrt{5}+\sqrt{3}}$

$$=\frac{35+12\sqrt{15}+15}{5-3}+\frac{35-12\sqrt{15}+15}{5-3}$$
$$=50.$$

15. If
$$a+b : b+c = c+d : d+a,$$
then
$$(a+b)(d+a) = (b+c)(c+d);$$
$$\therefore a^2 + a(b+d) + bd = c^2 + c(b+d) + bd;$$
$$\therefore a^2 - c^2 + (a-c)(b+d) = 0,$$
i.e.
$$(a-c)(a+b+c+d) = 0.$$
Hence
$$a = c \text{ or } a+b+c+d = 0.$$

16. The sum of all numbers from 1 to $1000 = \frac{1}{2} . 1000 . 1001 = 500500$.

(1) The multiples of 3 are 3, 6, 9, ..., 999, and the sum of these is
$$3(1+2+3+\ldots+333) = 3 . \frac{1}{2} . 333 . 334 = 166833.$$

Hence the sum of all the numbers which are not multiples of 3 is $500500 - 166833 = 333667$.

(2) The sum of all the even numbers is
$$2(1+2+3+\ldots+500) = 500 . 501 = 250500.$$
The sum of all the multiples of 5 is
$$5(1+2+\ldots+200) = 5 . \frac{1}{2} . 200 . 201 = 100500.$$
The sum of the multiples of 2×5 is
$$10(1+2+\ldots+100) = 10 . \frac{1}{2} . 100 . 101 = 50500.$$
Hence the sum of all numbers which are not divisible by 2 or by 5 is
$$500500 - 250500 - 100500 + 50500 = 200000.$$

17. The different ways of voting are shown as the coefficient of x^5 in the product
$$\begin{aligned}&(1 + ax + aax^2 + aaax^3 + aaaax^4 + \ldots)\\ &\times(1 + bx + bbx^2 + bbbx^3 + \ldots)\\ &\times(1 + cx + ccx^2 + cccx^3 + \ldots)\\ &\times(1 + dx + ddx^2 + dddx^3 + \ldots)\\ &\times(1 + ex + eex^2 + eeex^3 + \ldots).\end{aligned}$$
Hence the *number* of ways of voting is the coefficient of x^5 in
$$(1 + x + x^2 + x^3 + \ldots)^5,$$
that is in $(1-x)^{-5}$, and this coefficient is $\dfrac{5.6.7.8.9}{1.2.3.4.5} = 126$.

18. Any particular boy can have two prizes in $\frac{1}{2}(n+1)n$ ways; and when this boy has got his two prizes the $n-1$ remaining prizes can be given singly to the other boys in $\underline{|n-1}$ ways. Thus there are $\frac{1}{2}(n+1)\underline{|n}$ ways of distributing the prizes so that a particular boy may have two of them. Hence the whole number of different ways of giving the prizes is $\frac{1}{2}n\,\underline{|n+1}$.

19. $(1-x)(1+x)^{-2}=(1-x)\{1-2x+3x^2-\ldots+(-1)^r(r+1)x^r+\ldots\}$.
Hence the coefficient of x^n is $(-1)^n\{(n+1)+n\}$.

20.
$$\log\left(1-\frac{x}{1+x}\right)=-\left[\frac{x}{1+x}+\frac{1}{2}\left(\frac{x}{1+x}\right)^2+\ldots\right]$$

and
$$\log\left(1-\frac{x}{1+x}\right)=\log\frac{1}{1+x}=-\log(1+x).$$

21.
$$\begin{aligned}&(4a^2-2b^2-2c^2+5bc+2ca+2ab)\\&+(-2a^2+4b^2-2c^2+2bc+5ca+2ab)\\&+(-2a^2-2b^2+4c^2+2bc+2ca+5ab)\\&=9\Sigma bc.\end{aligned}$$

22. (i)
$$\begin{aligned}&(a+b+c)^2+a^2-b^2-c^2\\&=2a^2+2bc+2ca+2ab\\&=2(a+b)(a+c).\end{aligned}$$

(ii)
$$\begin{aligned}&1+y(1+x)^2(1+xy)\\&=1+y(1+x)^2+y^2x(1+x)^2\\&=\{1+y(1+x)\}\{1+yx(1+x)\}.\end{aligned}$$

23. (i)
$$\frac{ax}{b}-\frac{b}{ax}=b-\frac{1}{b};$$
$$\therefore\ a^2x^2-ax(b^2-1)-b^2=0,$$

i.e.
$$(ax+1)(ax-b^2)=0.$$

(ii)
$$x+y=1,\ x^3-xy+y^3=4.$$

Substitute for x; then
$$\begin{aligned}&(1-y)^3-y(1-y)+y^3=4;\\&\therefore\ 1-4y+4y^2=4;\\&\therefore\ 1-2y=\pm2.\end{aligned}$$

Hence
$$y=\frac{3}{2}\text{ and then }x=-\frac{1}{2},$$

or
$$y=-\frac{1}{2}\text{ and then }x=\frac{3}{2}.$$

24. The required equation is
$$\left(x-\alpha-\frac{2}{\beta}\right)\left(x-\beta-\frac{2}{\alpha}\right)=0,$$

i.e.
$$x^2-x\left(\alpha+\beta+\frac{2}{\alpha}+\frac{2}{\beta}\right)+\alpha\beta+4+\frac{4}{\alpha\beta}=0.$$

Now
$$\alpha+\beta=-p,$$
$$\alpha\beta=q,$$
$$\therefore\ \frac{1}{\alpha}+\frac{1}{\beta}=-\frac{p}{q} \text{ and } \frac{1}{\alpha\beta}=\frac{1}{q}.$$

Hence we have
$$x^2+x\left(p+\frac{2p}{q}\right)+q+4+\frac{4}{q}=0,$$
i.e.
$$qx^2+(pq+2p)\,x+(q+2)^2=0.$$

25. Let the trains from A and B travel at x and y miles per hour respectively; then they will meet in $\frac{100}{x+y}$ hours. Also they will take $\frac{100}{x}$ hours and $\frac{100}{y}$ hours to complete the journey. Let $x>y$.

Then
$$\frac{100}{x+y}=\frac{3}{2},$$
and
$$\frac{100}{x}-\frac{100}{y}=-\frac{75}{60}=-\frac{5}{4}.$$
$$\therefore\ x-y=\frac{1}{80}xy,$$
and
$$x+y=\frac{200}{3}.$$
$$\therefore\ 2x-\frac{200}{3}=\frac{1}{80}x\left(\frac{200}{3}-x\right),$$
whence
$$3x^2+280x-16000=0,$$
i.e.
$$(x-40)(3x+400)=0.$$

Hence $x=40$, since a negative value is in this case meaningless.

Also
$$y=\frac{200}{3}-40=26\tfrac{2}{3}.$$

26.
$$\frac{cy+bz}{l}=\frac{az+cx}{m}=\frac{bx+ay}{n}=\lambda \text{ suppose.}$$

Then
$$cy+bz=l\lambda.$$
$$\therefore\ \frac{y}{b}+\frac{z}{c}=al\,.\,\frac{\lambda}{abc}.$$

Also
$$\frac{z}{c}+\frac{x}{a}=bm\,.\,\frac{\lambda}{abc},$$
$$\frac{x}{a}+\frac{y}{b}=cn\,.\,\frac{\lambda}{abc}.$$

Subtract the first from the sum of the other two ; then

$$\frac{2x}{a}=(bm+cn-al)\frac{\lambda}{abc}.$$

Hence $$\frac{x}{a(bm+cn-al)}=\frac{\lambda}{2abc}=\&c.$$

27. $$\frac{\sqrt{3}}{\sqrt{5}+\sqrt{2}}-\frac{\sqrt{3}}{\sqrt{500}-\sqrt{200}}=\sqrt{3}\left\{\frac{1}{\sqrt{5}+\sqrt{2}}-\frac{1}{10(\sqrt{5}-\sqrt{2})}\right\}$$
$$=\sqrt{3}\left\{\frac{\sqrt{5}-\sqrt{2}}{3}-\frac{\sqrt{5}+\sqrt{2}}{30}\right\}=\frac{\sqrt{3}}{30}\{9\sqrt{5}-11\sqrt{2}\}.$$

And $$\frac{5}{\sqrt{15}+\sqrt{6}}-\frac{1}{\sqrt{60}-\sqrt{24}}=\frac{5(\sqrt{15}-\sqrt{6})}{9}-\frac{\sqrt{60}+\sqrt{24}}{36}$$
$$=\frac{1}{36}\{20\sqrt{15}-20\sqrt{6}-2\sqrt{15}-2\sqrt{6}\}$$
$$=\frac{1}{18}\{9\sqrt{15}-11\sqrt{6}\}=\frac{\sqrt{3}}{18}(9\sqrt{5}-11\sqrt{2}),$$
$$\therefore\ \text{Quotient}=\frac{18}{30}=\frac{3}{5}.$$

28. Let m be the first of the integers.

Then $$S_{2n+1}=\frac{2n+1}{2}(2m+2n)$$
$$=(m+n)(2n+1),$$
$$S_{2n}=\frac{2n}{2}(2m+2n-1).$$

Since $2m+2n-1$ is odd, S_{2n} is always divisible by n but never by $2n$.

29. From the terms containing a^3, b^3, c^3 we see that if the given expression is a perfect cube it must be $\left(2a-\frac{3b}{2}+\frac{c}{3}\right)^3$. Write down $\left(2a-\frac{3b}{2}+\frac{c}{3}\right)^3$, and it will be seen to be identical with the given expression.

30. $$\left(\frac{1-x}{1-3x}\right)^2=(1-2x+x^2)\{1+2\,.\,3x+3\,.\,3^2x^2+\ldots+(r+1)\,3^rx^r+\ldots\}.$$

Hence the coefficient of x^r
$$=(r+1)\,3^r-2r\,.\,3^{r-1}+(r-1)\,3^{r-2}$$
$$=3^{r-2}\{9(r+1)-6r+(r-1)\}$$
$$=3^{r-2}(4r+8).$$

31. (i) $$x^5-y^5-(x-y)^5$$
$$=x^5-y^5-(x^5-5x^4y+10x^3y^2-10x^2y^3+5xy^4-y^5)$$
$$=5xy(x^3-2x^2y+2xy^2-y^3)$$
$$=5xy(x-y)(x^2-xy+y^2).$$

(ii) Arrange in powers of x; then

$$x^2y(y+1)+x(2y^2+2y+1)+y(y+1)$$
$$=\{xy+y+1\}\{x(y+1)+y\}.$$

32.
$$F=\frac{1}{abc}\cdot\Sigma\frac{a^2}{(c-a)(a-b)}$$
$$=\frac{1}{abc(b-c)(c-a)(a-b)}\cdot\Sigma a^2(b-c)$$
$$=-\frac{(b-c)(c-a)(a-b)}{abc(b-c)(c-a)(a-b)}=-\frac{1}{abc}.$$

33. (i) Multiply by $a-b$; then

$$x^2(a-b)-a^2(x-b)+b^2(x-a)=0,$$

i.e.
$$(a-b)\{x^2-(a+b)x+ab\}=0,$$
$$\therefore\ (x-a)(x-b)=0.$$

(ii)
$$x^3-a^3-4(x-a)=0,$$

i.e.
$$(x-a)(x^2+ax+a^2-4)=0.$$

Hence
$$x=a \text{ or } x=-\frac{1}{2}a\pm\frac{1}{2}\sqrt{(16-3a^2)}.$$

(iii) This is the reciprocal equation

$$2x^4-3x^3+2x^2-3x+2=0,$$

i.e.
$$2\left(x^2+\frac{1}{x^2}\right)-3\left(x+\frac{1}{x}\right)+2=0.$$

Put
$$y=x+\frac{1}{x};\text{ then } x^2+\frac{1}{x^2}=y^2-2.$$

Hence
$$2(y^2-2)-3y+2=0,$$

i.e.
$$(2y+1)(y-2)=0.$$

If
$$x+\frac{1}{x}=2,\ (x-1)^2=0.$$

If
$$x+\frac{1}{x}=-\frac{1}{2},\ 2x^2+x+2=0,$$

and
$$\therefore\ x=\frac{1}{4}(-1\pm\sqrt{-15}).$$

Thus the four roots are

$$1,\ 1,\ \frac{1}{4}(-1+\sqrt{-15}),\ \frac{1}{4}(-1-\sqrt{-15}).$$

34. The condition for equal roots is

$$a^2-4(a+3)=0,$$

whence
$$a=6 \text{ or } a=-2.$$

35. The expression must be

$$\left(x^2+\frac{a}{2}x+\sqrt{n}\right)^2=x^4+ax^3+\left(\frac{a^2}{4}+2\sqrt{n}\right)x^2+a\sqrt{nx}+n.$$

Hence $$m=\frac{a^2}{4}+2\sqrt{n},$$

and $$c=a\sqrt{n}.$$

Thus $$n=\frac{c^2}{a^2} \text{ and } m=\frac{a^2}{4}+2\frac{c}{a}.$$

36. x is one of the imaginary cube roots of unity. Hence $x^3=1$ and $x^2+x+1=0$.

In the given expression substitute $x^3=1$ and $x^4=x\,.\,x^3=x$; then we have

$$x-5-12x^2-13x-7=-12(x^2+x+1)=0.$$

37. We have to prove that

$$b^2c^2+c^2a^2+a^2b^2-abc(a+b+c)>0,$$

i.e. $$\frac{1}{2}\{(bc-ca)^2+(ca-ab)^2+(ab-bc)^2\}>0.$$

38. Let the series be $\frac{1}{a}, \frac{1}{a+d}, \frac{1}{a+2d}, \ldots$. Then the mth, the nth and the $(m+n)$th terms will be respectively

$$\frac{1}{a+(m-1)d}, \quad \frac{1}{a+(n-1)d} \text{ and } \frac{1}{a+(m+n-1)d}.$$

Hence $$a+(m-1)d=\frac{1}{n},$$

and $$a+(n-1)d=\frac{1}{m}.$$

Hence $$(m-n)d=\frac{m-n}{mn}, \text{ and } \therefore d=\frac{1}{mn}.$$

Then $$a+(m-1)\frac{1}{mn}=\frac{1}{n}, \text{ and } \therefore a=\frac{1}{mn}.$$

Hence the $(m+n)$th term

$$=\frac{mn}{1+(m+n-1)}=\frac{mn}{m+n}.$$

39. $$(1-x)^{-\frac{1}{2}}=1+\frac{1}{2}x+\frac{1\,.\,3}{2\,.\,4}x^2+\frac{1\,.\,3\,.\,5}{2\,.\,4\,.\,6}x^3+\ldots.$$

Hence $$\left(1-\frac{3}{4}\right)^{-\frac{1}{2}}=1+\frac{3}{2^3}+\frac{1\,.\,3}{1\,.\,2}\cdot\frac{3^2}{2^6}+\frac{1\,.\,3\,.\,5}{1\,.\,2\,.\,3}\frac{3^3}{2^9}+\ldots.$$

Thus the series $$=\left(1-\frac{3}{4}\right)^{-\frac{1}{2}}=4^{\frac{1}{2}}=2.$$

40.
$$\log \sqrt[5]{(18 \times \cdot 0015)} = \frac{1}{5} \log (27 \times \cdot 001)$$
$$= \frac{1}{5} (3 \log 3 - 3) = \frac{1}{5} (-1\cdot 5686361) = -\cdot 3137272$$
$$= \bar{1}\cdot 6862728.$$

Now $\log \cdot 48559 = \bar{1}\cdot 6862697,$

and $\log \cdot 48560 = \bar{1}\cdot 6862787$, and $\frac{31}{90} = \cdot 3.$

Hence the approximate root is $\cdot 485593$.

41. The numerator
$$= \left(\frac{1}{a} + \frac{1}{b}\right)^2 - \frac{(a-b)^2 + 4ab}{ab\,(a-b)^2}$$
$$= \frac{(a+b)^2}{a^2b^2} - \frac{(a+b)^2}{ab\,(a-b)^2}$$
$$= \left(\frac{a+b}{a-b}\right)^2 \left\{\frac{(a-b)^2}{a^2b^2} - \frac{1}{ab}\right\} = \left(\frac{a+b}{a-b}\right)^2 \left\{\left(\frac{1}{a} - \frac{1}{b}\right)^2 - \frac{1}{ab}\right\}.$$

Hence the fraction $= \left(\frac{a+b}{a-b}\right)^2.$

42. From the first two equations
$$ax - y + z = 0,$$
and
$$x + by - z = 0,$$
we have
$$\frac{x}{1-b} = \frac{y}{1+a} = \frac{z}{ab+1}.$$

Substitute in the third equation; then
$$1 - b - (1+a) = c\,(1 + ab);$$
$$\therefore\ a + b + c + abc = 0.$$

43. (i)
$$\frac{2x - y}{10a + 3b} = \frac{x - 3y}{4b},$$
$$\therefore\ \text{each} = \frac{2x - y - 2\,(x - 3y)}{10a + 3b - 2\,.\,4b} = \frac{y}{2a - b}.$$

Thus
$$\frac{y+b}{2a} = \frac{y}{2a-b};$$
$$\therefore\ y = 2a - b.$$

Then
$$\frac{2x - y}{10a + 3b} = \frac{y + b}{2a} = 1;$$
$$\therefore\ 2x = y + 10a + 3b = 12a + 2b;$$
$$\therefore\ x = 6a + b.$$

(ii) $$2x^2+2x-1=2(x+1)\sqrt{(x^2-1)};$$
$$\therefore\ (2x^2+2x-1)^2=4(x+1)^2(x^2-1);$$
$$\therefore\ 4x^4+8x^3-4x+1=4(x^4+2x^3-2x-1).$$

Hence $$4x+5=0.$$

44. Let x tons be bought at y shillings per ton.

Then 1 cwt. sold for y pence;

$\therefore$ 1 ton......... $20y$ pence.

Hence profit per ton $=\left(\frac{20y}{12}-y\right)$ shillings;

$\therefore$ whole profit $=\frac{8}{12}xy$ shillings $=\frac{8}{240}xy$ pounds.

Hence $$\frac{8}{240}xy=y,\text{ and }\therefore\ x=30.$$

45. Having arranged in powers of x, we have
$$4x^2a^{-2}-12xa^{-1}+25-24x^{-1}a+16x^{-2}a^2.$$
The square root, namely $2xa^{-1}-3+4x^{-1}a$, can now be seen by inspection.

46. $$x:y=1\tfrac{1}{2}:2\tfrac{2}{3}=9:16.$$
Also $$x+15:y+15=1\tfrac{2}{3}:2\tfrac{1}{2}=2:3.$$
$$\therefore\ 16x=9y,\text{ and }3x+45=2y+30,$$
whence $$x=27\text{ and }y=48.$$

47. Since x is real,
$$b^2(a+c)^2-(a^2+b^2)(b^2+c^2)\not<0,$$
i.e. $$2acb^2-a^2c^2-b^4\not<0,$$
i.e. $$-(ac-b^2)^2\not<0.$$
Hence $$ac-b^2=0.$$

48. (i) $$1.3+3.5+\ldots+(2n-1)(2n+1).$$
$$(2n-1)(2n+1)=\frac{1}{6}\{(2n-1)(2n+1)(2n+3)-(2n-3)(2n-1)(2n+1)\}$$
$$\ldots\ldots\ldots\ldots\ldots\ldots=\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$
$$5.7=\frac{1}{6}\{5.7.9-3.5.7\}$$
$$3.5=\frac{1}{6}\{3.5.7-1.3.5\}$$
and $$1.3=\frac{1}{6}\{1.3.5+1.3\}.$$

Hence sum $$=\frac{1}{6}(2n-1)(2n+1)(2n+3)+\frac{1}{6}.1.3.$$

(ii)
$$\frac{1}{2\,.\,3}+\frac{1}{3\,.\,4}+\ldots+\frac{1}{(n+1)\,(n+2)}.$$
$$\frac{1}{2\,.\,3}=\frac{1}{2}-\frac{1}{3},$$
$$\frac{1}{3\,.\,4}=\frac{1}{3}-\frac{1}{4},$$
$$\ldots\ldots=\ldots\ldots$$
$$\frac{1}{(n+1)\,(n+2)}=\frac{1}{n+1}-\frac{1}{n+2}.$$

Hence
$$S_n=\frac{1}{2}-\frac{1}{n+2},\text{ and } S_\infty=\frac{1}{2}.$$

49.
$$\frac{(1+x)^{\frac{1}{2}}+(1-x)^{\frac{2}{3}}}{1+x+(1+x)^{\frac{1}{2}}}=\frac{1+\frac{1}{2}x+\left(1-\frac{2}{3}x\right)}{1+x+1+\frac{1}{2}x},$$

x^2 and higher powers of x are omitted, since x is very small,
$$=\frac{2-\frac{1}{6}x}{2+\frac{3}{2}x}=\frac{1}{2}\left(2-\frac{1}{6}x\right)\left(1+\frac{3}{4}x\right)^{-1}$$
$$=\frac{1}{2}\left(2-\frac{1}{6}x\right)\left(1-\frac{3}{4}x\right)$$
$$=\frac{1}{2}\left(2-\frac{5}{3}x\right)=1-\frac{5}{6}x.$$

50.
$$\log_e\left(1-\frac{2x^2}{1+x}\right)=\log_e\frac{1+x-2x^2}{1+x}=\log_e\frac{(1+2x)\,(1-x)}{1+x}$$
$$=\log_e(1+2x)+\log_e(1-x)-\log_e(1+x)$$
$$=2x-\frac{1}{2}(2x)^2+\frac{1}{3}(2x)^3-\frac{1}{4}(2x)^4+\frac{1}{5}(2x)^5-\ldots$$
$$-x-\frac{1}{2}x^2-\frac{1}{3}x^3-\frac{1}{4}x^4-\frac{1}{5}x^5-\ldots$$
$$-x+\frac{1}{2}x^2-\frac{1}{3}x^3+\frac{1}{4}x^4-\frac{1}{5}x^5+\ldots$$
$$=-2x^2+2x^3-4x^4+6x^5.$$

51.
$$\frac{(a+b)\,(a+c)}{(a+b)\,(a+2c)}\times\frac{(a+2c)\,(a^2-2ac+4c^2)}{(a+c)\,(a^3-a^2c+2ac^2+4c^3)}=\frac{1}{a+c}.$$

52. The expression is
$$x^2a\,(b+c)+xy\,(b^2-c^2-a^2)+y^2a\,(c-b)$$
$$=\{ax-y\,(c-b)\}\,\{(b+c)\,x-ay\}.$$

53. (i) From
$$ax+y=x+by,$$
$$x(a-1)=y(b-1).$$

Substitute for x in
$$2x+2by=x+y+2,$$
$$2y(b-1)+2by(a-1)=y(b-1)+y(a-1)+2(a-1),$$
$$\therefore\ y=2(a-1)/(2ab-a-b),$$
and
$$\therefore\ x=2(b-1)/(2ab-a-b).$$

(ii) We have
$$(3x-6)(10-4x)+(11-2x)(5-x)=7(5-2x)(5-x);$$
whence
$$24x^2-138x+180=0;$$
$$\therefore\ 6(4x-15)(x-2)=0.$$

(iii) We have
$$\frac{x^3}{y}(1+xy)=108 \ldots\ldots(\alpha),$$
and
$$\frac{y^3}{x}(1+xy)=\frac{4}{3} \ldots\ldots(\beta).$$

From (α) and (β) by multiplication
$$x^2y^2(1+xy)^2=144;$$
$$\therefore\ xy(1+xy)=\pm 12.$$
$$\therefore\ xy=3,\quad xy=-4,$$
or
$$xy=\frac{1}{2}(-1\pm\sqrt{-47}).$$

From (α) and (β) by division $\dfrac{x^4}{y^4}=81;$
$$\frac{x^2}{y^2}=\pm 9,$$
and
$$\therefore\ x=\pm 3y \text{ or } x=\pm 3y\sqrt{-1}.$$
From
$$xy=3 \text{ and } x=\pm 3y,$$
we have
$$x=\pm 3,\quad y=\pm 1,$$
or
$$x=\pm 3\sqrt{-1},\quad y=\mp 3\sqrt{-1}.$$
Similarly for the other roots.

54. A walks $\dfrac{m}{n}$ miles per hour, and B walks $\dfrac{18n}{m}$ miles per hour.

Hence
$$\frac{m}{n}-\frac{18n}{m}=\pm\frac{1}{2};$$
$$\therefore\ 2\left(\frac{m}{n}\right)^2-\frac{m}{n}-18=0,$$
or
$$2\left(\frac{m}{n}\right)^2+\frac{m}{n}-18=0.$$

From (I) $\frac{m}{n}=4\frac{1}{2}$ (rejecting the negative value) and $\therefore$ A walks at $4\frac{1}{2}$ miles an hour and B at 4 miles an hour.

From (II) $\frac{m}{n}=4=A$'s rate and $4\frac{1}{2}=B$'s rate.

55. $[(\sqrt{a}+\sqrt{d})(\sqrt{b}+\sqrt{c})][(\sqrt{a}-\sqrt{d})(\sqrt{b}-\sqrt{c})]=(a-d)(b-c)$.

56. (i) $x^2-2x\left(\frac{b}{2}-\frac{a}{3}\right)+\frac{b^2}{4}-\frac{ab}{3}+\frac{a^2}{9}=\left\{x-\left(\frac{b}{2}-\frac{a}{3}\right)\right\}^2$.

(ii) $\frac{9}{4}+6x-17x^2-28x^3+49x^4=\left(\frac{3}{2}+2x-7x^2\right)^2$.

(iii) $x^4-x^3y\sqrt{2}+\frac{5}{2}x^2y^2-xy^3\sqrt{2}+y^4=\left(x^2-\frac{1}{\sqrt{2}}xy+y^2\right)^2$.

57. If the series is a, $a+d$, $a+2d$, &c., we have

$$\frac{10}{2}(2a+9d):\frac{5}{2}(2a+4d)=13:4,$$

whence $$a=2d.$$

Again, $$115=\frac{20}{2}(2a+19d);$$

$$\therefore\ 20a+190d=115.$$

Hence $$a=2d=1.$$

58. $$\frac{a+c}{ac}=\frac{4b-a-c}{4b^2-2b(a+c)+ac};$$

$$\therefore\ 4b^2(a+c)-2b\{(a+c)^2+2ac\}+2ac(a+c)=0;$$

$$\therefore\ \{2b-(a+c)\}\{b(a+c)-ac\}=0;$$

$$\therefore\ b=\frac{1}{2}(a+c),$$

or $$2b=\frac{2ac}{a+c}.$$

59. The ratio is $$\frac{(n-4)(n-5)}{5.6}3^2x^2:1;$$

$$\therefore\ \frac{(n-4)(n-5)}{5.6}3^2=\frac{7}{3};$$

$$\therefore\ 9n^2-81n+110=0,$$

i.e. $$(3n-22)(3n-5)=0;$$

$$\therefore\ n=\frac{22}{3},\ \text{or}\ n=\frac{5}{3}.$$

60. If the square root be found in the ordinary way, we find that $(x^2-3x+2)^2$ agrees with the given expression so far as the first three terms are concerned, but differs from it by $4x+12$. The given expression is therefore a perfect square when $x=-3$.

Again, $(x^2-3x+4)^2=x^4-6x^3+14x^2-12x+16$.

Hence given expression will be a square if $x^2-4x=0$, *i.e.* if $x=0$ or $x=4$.

Again, $(x^2-x+4)^2=x^4-2x^3+9x^2-8x+16$.

This is equivalent to the given expression when $4x^3-4x^2=0$, *i.e.* when $x=0$ or $x=1$.

Hence given expression is a perfect square for any one of the values 0, 1, 4 or -3.

61.
$$\begin{aligned} N&=1-\Sigma a^2+\Sigma a^2b^2-a^2b^2c^2+(abc+abc\Sigma a^2-\Sigma b^2c^2-a^2b^2c^2)\\ &=1+abc-2a^2b^2c^2-\Sigma a^2(1-abc)\\ &=(1-abc)\{1+2abc-\Sigma a^2\}.\end{aligned}$$

$\therefore$ fraction $=1+2abc-a^2-b^2-c^2$.

62.
$$\begin{aligned}\text{The expression}&=2(x^2+y^2+z^2)-2yz-2zx-2xy\\ &=(y-z)^2+(z-x)^2+(x-y)^2.\end{aligned}$$

63. $a^3+b^3+c^3-3abc=\frac{1}{2}(a+b+c)\{(b-c)^2+(c-a)^2+(a-b)^2\}$.

Hence, if $a^3+b^3+c^3-3abc=0$,

either $a=b=c$ or $a+b+c=0$.

If $a=b=c$, then $a^2-bc=0=b^2-ca=c^2-ab$.

Also, if $a+b+c=0$,

$$a^2-bc=a^2+b(a+b)=b^2+a(a+b)=b^2-ac.$$

Similarly $b^2-ac=c^2-ab$.

64. (i) $\sqrt{(5x-14)}-\sqrt{(2x+1)}=2\sqrt{x}-\sqrt{(3x-13)}$.

Square both sides; then

$$5x-14+2x+1-2\sqrt{(5x-14)}\sqrt{(2x+1)}=4x+3x-13-4\sqrt{x}\sqrt{(3x-13)};$$
$$\therefore \sqrt{(5x-14)}\sqrt{(2x+1)}=2\sqrt{x}\sqrt{(3x-13)};$$
$$\therefore 10x^2-23x-14=12x^2-52x;$$
$$\therefore 2x^2-29x+14=0,$$

i.e. $(2x-1)(x-14)=0$.

(ii) $2ax-by=a^2$ and $bxy=a^3$.

Hence $2ax^2-a^3=a^2x$,

whence $x=a$, or $x=-\frac{a}{2}$.

And, if $x=a$, $y=a^2/b$;

if $x=-\frac{a}{2}$, $y=-2a^2/b$.

(iii) $x+y=12, \quad x^4+y^4=3026.$

$x^4+y^4=(x+y)^4-4x^3y-6x^2y^2-4xy^3;$

$\therefore \; 12^4-2xy\,(2x^2+3xy+2y^2)=3026;$

$\therefore \; xy\{2\,(x+y)^2-xy\}=8855;$

$\therefore \; x^2y^2-288xy+8855=0,$

whence $xy=144\pm\sqrt{(144^2-8855)}=144\pm109.$

I. If $xy=35$ and $x+y=12$,

$x=5, \quad y=7,$

or $x=7, \quad y=5.$

II. If $xy=253$ and $x+y=12$,

$x=6\pm\sqrt{-217}, \quad y=6\mp\sqrt{-217}.$

65. We have $(x+2y)(a+4b)=(a+3b)\,(y+3x);$

$\therefore \; y\,(a+5b)=x\,(2a+5b);$

$\therefore \; \dfrac{x}{y}=\dfrac{a+5b}{2a+5b}.$

Hence $\dfrac{y+2x}{x+3y}=\dfrac{2a+5b+2\,(a+5b)}{a+5b+3\,(2a+5b)}=\dfrac{4a+15b}{7a+20b}.$

66. $3\,(c-1)=\sqrt[3]{100}+2\sqrt[3]{10}=10^{\frac{2}{3}}+2\,.\,10^{\frac{1}{3}};$

$\therefore \; 27\,(c-1)^3=10^2+6\,.\,10^{\frac{5}{3}}+12\,.\,10^{\frac{4}{3}}+8\,.\,10$

$=180+60\,(10^{\frac{2}{3}}+2\,.\,10^{\frac{1}{3}})$

$=180+60\times3\,(c-1)=180c;$

$\therefore \; 3\,(c-1)^3=20c.$

67. Put $x+\dfrac{9}{x+2}=y.$

Then $x^2+x\,(2-y)+9-2y=0.$

Since x is real, $(2-y)^2-4\,(9-2y)\nless 0,$

i.e. $y^2+4y-32\nless 0,$

i.e. $(y+8)(y-4)\nless 0.$

$\therefore \; y>4$ or $y<-8.$

68. $x^2y^2z^2\,(x^{-3}+y^{-3}+z^{-3})-x^3-y^3-z^3$

$=\dfrac{1}{xyz}\{y^3z^3+z^3x^3+x^3y^3-xyz\,(x^3+y^3+z^3)\}$

$=\dfrac{1}{xyz}\,(yz-x^2)(zx-y^2)(xy-z^2)$

$=0$, if x, y, z are in G.P. in any order.

Or thus: If x, y, z are in G.P., $y^2=xz.$

Hence $x^2y^2z^2\,(x^{-3}+y^{-3}+z^{-3})=x^3z^3\,(x^{-3}+z^{-3})+y^6\times y^{-3}$

$=x^3+y^3+z^3.$

69. Multiply by x^4+1; then

$$x^2+2=(A+Bx)(x^2-x\sqrt{2}+1)+(C+Dx)(x^2+x\sqrt{2}+1);$$

$$\therefore\ (B+D)x^3+(A+C-B\sqrt{2}+D\sqrt{2}-1)x^2$$
$$+(B+D-A\sqrt{2}+C\sqrt{2})x+A+C-2=0.$$

If this is true for *all values* of x, we must have

$$B+D=0,\ A+C-B\sqrt{2}+D\sqrt{2}-1=0,$$
$$B+D-\sqrt{2}(A-C)=0 \text{ and } A+C-2=0.$$

Hence $$A=C=1,\ B=-D=\frac{1}{4}\sqrt{2}.$$

70. Neglecting powers of x above the third in

$$\frac{(1-2x)^{-\frac{1}{2}}-(1+2x)^{\frac{1}{2}}}{(1-x)^{-\frac{1}{2}}-(1+x)^{\frac{1}{2}}},$$

$$N=1+\frac{1}{2}\cdot 2x+\frac{1\,.\,3}{2\,.\,4}\cdot 4x^2+\frac{1\,.\,3\,.\,5}{2\,.\,4\,.\,6}\cdot 8x^3+\ldots$$
$$-\left(1+\frac{1}{2}\cdot 2x-\frac{1\,.\,1}{2\,.\,4}4x^2+\frac{1\,.\,1\,.\,3}{2\,.\,4\,.\,6}\cdot 8x^3+\ldots\right).$$

$$D=1+\frac{1}{2}x+\frac{1\,.\,3}{2\,.\,4}x^2+\frac{1\,.\,3\,.\,5}{2\,.\,4\,.\,6}x^3-\left(1+\frac{1}{2}x-\frac{1\,.\,1}{2\,.\,4}x^2-\frac{1\,.\,1\,.\,3}{2\,.\,4\,.\,6}x^3\right).$$

Hence fraction $$=\frac{2x^2+2x^3}{\frac{1}{2}x^2+\frac{1}{4}x^3}=\frac{4+4x}{1+\frac{1}{2}x}=(4+4x)\left(1-\frac{1}{2}x\right)$$
$$=4+2x.$$

71. Five consecutive integers are

$$n-2,\ n-1,\ n,\ n+1,\ n+2.$$

Hence the sum of their squares

$$=(n-2)^2+(n-1)^2+n^2+(n+1)^2+(n+2)^2$$
$$=5n^2+4+1+1+4$$
$$=5(n^2+2).$$

Again $$(n-3)^2+(n-2)^2+(n-1)^2+n^2+(n+1)^2+(n+2)^2+(n+3)^2$$
$$=7n^2+9+4+1+1+4+9$$
$$=7(n^2+4).$$

72. $$x-a=b+\frac{(a-b)^2}{4(a+b)}=\frac{(a+b)^2+4b^2}{4(a+b)}=\frac{1}{4}(a+b)+\frac{b^2}{a+b},$$

$$y-b=\frac{1}{4}(a+b)+\frac{ab}{a+b}-b=\frac{1}{4}(a+b)-\frac{b^2}{a+b}.$$

Hence $$(x-a)^2-(y-b)^2=\frac{1}{16}(a+b)^2+\frac{1}{2}b^2+\frac{b^4}{(a+b)^2}$$
$$-\left\{\frac{1}{16}(a+b)^2-\frac{1}{2}b^2+\frac{b^4}{(a+b)^2}\right\}$$
$$=b^2.$$

73. (i) We have $$1+\frac{1}{x-2}+3+\frac{1}{x-3}=4+\frac{9}{x+2};$$
$$\therefore\ \frac{1}{x-2}+\frac{1}{x-3}=\frac{9}{x+2};$$
$$\therefore\ (x-3)(x+2)+(x+2)(x-2)=9(x-2)(x-3);$$
$$\therefore\ 7x^2-44x+64=0,$$
i.e. $$(x-4)(7x-16)=0.$$

(ii) From the first equation subtract four times the second; then
$$(x-y)^2-9(x-y)+14=0,$$
i.e. $$(x-y-2)(x-y-7)=0.$$
If $$x-y=2,\ (x+y)^2+6=30;$$
$$\therefore\ x+y=\pm 2\sqrt{6}.$$
Hence $$x=1\pm\sqrt{6} \text{ and } y=-1\pm\sqrt{6}.$$
If $$x-y=7,\ (x+y)^2+21=30;$$
$$\therefore\ x+y=\pm 3.$$
Hence $$x=5 \text{ and } y=-2,$$
or $$x=2 \text{ and } y=-5.$$

74. Let C be the capital, and x per cent. the rate of interest.

Then $$C\cdot\frac{x}{100}=100,$$
and $$(C+500)\frac{x-1}{100}=100.$$
Hence $$(C+500)(Cx-C)=10000C;$$
$$\therefore\ (C+500)(10000-C)=10000C;$$
$$\therefore\ C^2+500C-5000000=0,$$
i.e. $$(C-2000)(C+2500)=0.$$
Hence the sum invested was £2000.

75. Multiply by x^6; then, since $xy=x^2+1$, we have
$$(x^2+1)^6+x(x^2+1)^5-5x^2(x^2+1)^4-4x^3(x^2+1)^3$$
$$+6x^4(x^2+1)^2+3x^5(x^2+1)-x^6=0.$$
Arrange in powers of x, and we have
$$x^{12}+x^{11}+x^{10}+x^9+x^8+x^7+x^6+x^5+x^4+x^3+x^2+x+1=0,$$
i.e. $$(x^{13}-1)/(x-1)=0.$$

76.
$$\left(\frac{x+a}{x-a}\right)^2-\frac{x}{3a}=\left\{\frac{2a\,(1+\sqrt{3})}{2a\sqrt{3}}\right\}^2-\frac{1}{3}(1+2\sqrt{3})$$
$$=\frac{1}{3}(4+2\sqrt{3})-\frac{1}{3}(1+2\sqrt{3})=1.$$

77. We have
$$\frac{1}{2}(x+y)=\sqrt{(xy)}+2,$$
and
$$\frac{2xy}{x+y}=\frac{1}{5}x.$$
Hence
$$10xy=x^2+xy;$$
$$\therefore\ x=9y\ \text{(rejecting } x=0).$$

Hence, from the first equation, we get $y=1$. Thus the numbers are 9 and 1.

78.
$$\frac{3x-1}{6-x-x^2}=\frac{1}{2-x}-\frac{2}{3+x}$$
$$=\frac{1}{2}\left(1-\frac{x}{2}\right)^{-1}-\frac{2}{3}\left(1+\frac{x}{3}\right)^{-1}$$
$$=\frac{1}{2}\left(1+\frac{x}{2}+\frac{x^2}{4}+\dots+\frac{x^r}{2^r}+\dots\right)$$
$$-\frac{2}{3}\left(1-\frac{x}{3}+\frac{x^2}{3^2}-\dots+(-1)^r\frac{x^r}{3^r}\right)+\dots.$$

Hence the coefficient of x^n is
$$\frac{1}{2^{n+1}}+(-1)^{n+1}\cdot\frac{2}{3^{n+1}}.$$

79. Put
$$4x+\frac{9}{x}=y\ ;$$
then
$$4x^2-xy+9=0.$$

Since x is real,
$$y^2-4\,.\,4\,.\,9\nless 0,$$
i.e.
$$(y-12)\,(y+12)\nless 0.$$

Hence, if y is positive, it cannot be less than 12.

Again, put
$$\frac{x+2}{4x^2+16x+25}=y\ ;$$
then
$$4yx^2+(16y-1)\,x+25y-2=0.$$

Since x is real,
$$(16y-1)^2-4\,.\,4y\,.\,(25y-2)\nless 0,$$
i.e.
$$1-144y^2\nless 0.$$

Hence y cannot be greater than $\frac{1}{12}$.

80. $$1^5+1^7=2=\frac{1}{8}\,.\,1^4\,.\,2^4.$$

Thus the theorem is true when $n=1$.

Assume that $$S_{2n}=\frac{1}{8}\,n^4\,(n+1)^4.$$

Then $$\begin{aligned} S_{2\,(n+1)}&=\frac{1}{8}\,.\,n^4(n+1)^4+(n+1)^5+(n+1)^7\\ &=\frac{1}{8}\,(n+1)^4\,\{n^4+8\,(n+1)+8\,(n+1)^3\}\\ &=\frac{1}{8}\,(n+1)^4\,\{n^4+8n^3+24n^2+32n+16\}\\ &=\frac{1}{8}\,(n+1)^4\,(n+2)^4. \end{aligned}$$

Thus the theorem is true for $2\,(n+1)$ terms if it is true for $2n$. But it is true for 2, and $\therefore$ for 4, and $\therefore$ for 6; and so on.

81. $$(c-a)^5+(a-b)^5=\{(c-a)+(a-b)\}^5$$
$$-\{5\,(c-a)^4(a-b)+10\,(c-a)^3\,(a-b)^2+10\,(c-a)^2\,(a-b)^3+5\,(c-a)\,(a-b)^4\},$$
$$\therefore\ (b-c)^5+(c-a)^5+(a-b)^5$$
$$\begin{aligned} &=-5\,(c-a)\,(a-b)\,\{(c-a)^3+2\,(c-a)^2\,(a-b)+2\,(c-a)\,(a-b)^2+(a-b)^3\}\\ &=-5\,(c-a)\,(a-b)\,\{(c-a)+(a-b)\}\{(c-a)^2+(c-a)\,(a-b)+(a-b)^2\}\\ &=5\,(b-c)\,(c-a)\,(a-b)\,(a^2+b^2+c^2-bc-ca-ab). \end{aligned}$$

Or thus: Put $b=c$ and the given expression vanishes; $\therefore b-c$ is a factor. Similarly $c-a$ and $a-b$ are factors. The given expression is of the *fifth* degree, the remaining factor must therefore be of the second degree and symmetrical in a, b and c.

Thus $\Sigma\,(b-c)^5=(b-c)\,(c-a)\,(a-b)\,\{L\,(a^2+b^2+c^2)+M\,(bc+ca+ab)\}$, where L and M are numerical.

Comparing the coefficients of b^4c we have $-5=-L$. Also comparing the coefficients of b^3c^2 we have $10=L-M$. Hence $L=5=-M$.

82. We have
$$\begin{aligned} &(x+y+z)\,(x+y-z)\,(z+x-y)\,(z-x+y)\\ &=\{(x+y)^2-z^2\}\,z^2-(x-y)^2\\ &=2xy\times 2xy,\ \text{since}\ z^2=x^2+y^2\\ &=4x^2y^2. \end{aligned}$$

83. (i) $$(x+2)\,(x+1)-(x-1)\,(x-2)=\frac{3x}{44}\,(x-2)\,(x+1),$$

i.e. $$6x\times 44=3x\,(x-2)\,(x+1);$$
$$\therefore\ x=0,$$

or $$88=(x-2)(x+1),$$
whence $$x^2-x-90=0,$$
i.e. $$(x-10)(x+9)=0.$$

Thus the roots are 0, 10, −9.

(ii) $$\frac{5}{(x-2)(x-3)}-\frac{x}{(x-3)(x-4)}=\frac{4x+30}{(x-2)(x-4)};$$
$$\therefore\ 5(x-4)-x(x-2)=(4x+30)(x-3);$$
$$\therefore\ 5x^2+11x-70=0,$$
i.e. $$(5x-14)(x+5)=0.$$

(iii) $$x(y+y^2)=12,$$
$$x(1+y^3)=18;$$
$$\therefore\ \frac{y(1+y)}{(1+y)(1-y+y^2)}=\frac{12}{18}=\frac{2}{3};$$
$$\therefore\ 3y=2(1-y+y^2). \qquad [\text{If } y+1=0,\ x=\infty.]$$
$$\therefore\ 2y^2-5y+2=0,$$
i.e. $$(2y-1)(y-2)=0.$$

Hence $$y=2, \text{ and then } x=2,$$
or $$y=\frac{1}{2}, \text{ and then } x=16.$$

84. Let the man walk at x miles per hour.

Then $$\frac{15\frac{3}{4}}{x-1}=\frac{15\frac{3}{4}}{x}+1;$$
$$\therefore\ \frac{63}{4}\ \frac{1}{x(x-1)}=1;$$
$$\therefore\ 4x^2-4x-63=0,$$
i.e. $$(2x+7)(2x-9)=0.$$

Hence $x=4\frac{1}{2}$, for the negative value is meaningless.

Thus the rates are $4\frac{1}{2}$ and $3\frac{1}{2}$ miles per hour.

85. $$S_{50}=\frac{50}{2}\left\{34+49\left(-\frac{2}{7}\right)\right\}$$
$$=500,$$

and S_∞ for the series $200+120+72+\ldots$
$$=\frac{200}{1-\frac{3}{5}}=500.$$

86. The H.M. between a and c is $\frac{2ac}{a+c}$, so that $b=\frac{ac}{a+c}$.

$$\begin{aligned}a^3-b^3+c^3+3abc\\ &=(a-b+c)\,(a^2+b^2+c^2+bc-ca+ab)\\ &=(a-b+c)\,\{a^2+b^2+c^2+b\,(c+a)-ca\}\\ &=(a-b+c)\,(a^2+b^2+c^2)\\ &=(a-b+c)\,\{a^2+b^2+c^2-2b\,(a+c)+2ac\}\\ &=(a-b+c)^3.\end{aligned}$$

87. $$a^{-2}+2a^{-1}\,(2-b^{-2})+(2-b^{-2})^2=(a^{-1}+2-b^{-2})^2.$$

88. (i) $$\sqrt{\left\{\frac{\sqrt{20}+\sqrt{12}}{\sqrt{5}-\sqrt{3}}\right\}}=\sqrt{\left\{\frac{2\,(\sqrt{5}+\sqrt{3})}{\sqrt{5}-\sqrt{3}}\right\}}$$
$$=\sqrt{\left\{\frac{2\,(\sqrt{5}+\sqrt{3})^2}{5-3}\right\}}=\sqrt{5}+\sqrt{3}.$$

(ii) $$\begin{aligned}&\sqrt{\{3x-1+2\sqrt{(2x^2+x-6)}\}}\\ &=\sqrt{\{3x-1+2\sqrt{(2x-3)}\sqrt{(x+2)}\}}\\ &=\sqrt{\{2x-3+x+2+2\sqrt{(2x-3)}\sqrt{(x+2)}\}}\\ &=\sqrt{(2x-3)}+\sqrt{(x+2)}.\end{aligned}$$

89. $$(y-z)^2+(z-x)^2+(x-y)^2>0, \text{ unless } x=y=z.$$
Hence $$2\Sigma x^2>2\Sigma yz.$$

90. Let $$u_r=(3r+1)\,7^r-1.$$
Then $$u_{n+1}=(3n+4)\,7^{n+1}-1,$$
and $$u_n=(3n+1)\,7^n-1;$$
$$\begin{aligned}\therefore\ u_{n+1}-u_n&=7^n\,\{(3n+4)\,7-(3n+1)\}\\ &=7^n\,\{18n+27\}\\ &=\text{multiple of } 9.\end{aligned}$$

Hence u_{n+1} is a multiple of 9 if u_n is a multiple of 9.

But $u_1=(3+1)\,7-1=$multiple of 9, whence the theorem.

91. (i) $$\begin{aligned}x^4+4y^4&=(x^2+2y^2)^2-4x^2y^2\\ &=(x^2+2xy+2y^2)\,(x^2-2xy+2y^2).\end{aligned}$$

(ii) $$\begin{aligned}&(a^2+b^2)\,\{2\,(a^4-a^2b^2+b^4)-ab\,(2ab-3a^2+3b^2)\}\\ &=(a^2+b^2)\,\{2a^4+3a^3b-4a^2b^2-3ab^3+2b^4\}\\ &=(a^2+b^2)\,\{2\,(a^2-b^2)^2+3ab\,(a^2-b^2)\}\\ &=(a^2-b^2)\,(a^2+b^2)\,\{2a^2+3ab-2b^2\}\\ &=(a-b)\,(a+b)\,(a^2+b^2)\,(2a-b)\,(a+2b).\end{aligned}$$

92. Arrange in powers of a; then we have
$$\begin{aligned}&2a^3bc-a^2\,(b^3+c^3)-2ab^2c^2+bc\,(b^3+c^3)\\ &=2abc\,(a^2-bc)-(a^2-bc)\,(b^3+c^3)\\ &=(a^2-bc)\,(2abc-b^3-c^3)\\ &=0, \text{ since } a^2-bc=0.\end{aligned}$$

93. (i) Multiply by $\sqrt{(1+2x)}$; then

$$1+2x+\sqrt{(1-4x^2)}=\frac{12}{5};$$

$$\therefore\ (10x-7)^2=25\,(1-4x^2);$$

$$\therefore\ 200x^2-140x+24=0;$$

$$\therefore\ (10x-3)\,(5x-2)=0.$$

(ii) $$x^2+3xy=12-xy,$$

and $$x^2+3xy=16y^2-xy-x^2.$$

Hence $$2x^2+4xy-16y^2=0;$$

$$\therefore\ (x+4y)\,(x-2y)=0.$$

If $x=2y$, $$4y^2+6y^2=12-2y^2;$$

$$\therefore\ y=\pm 1, \text{ and } \therefore\ x=\pm 2.$$

If $x=-4y$, $$16y^2-12y^2=12+4y^2;$$

$$\therefore\ y=\infty.$$

(iii) $$x^3+8y^3=7xy=-7;$$

$$\therefore\ x^6-8=-7x^3;$$

i.e. $$(x^3-8)\,(x^3+1)=0.$$

If $x^3-8=0$, $$x=2 \text{ or } 2\omega \text{ or } 2\omega^2,$$

and $$\therefore\ y=-\frac{1}{2} \text{ or } -\frac{1}{2}\omega^2 \text{ or } -\frac{1}{2}\omega.$$

If $x^3+1=0$, $$x=-1 \text{ or } -\omega \text{ or } -\omega^2,$$

and $$\therefore\ y=1 \text{ or } +\omega^2 \text{ or } \omega.$$

In the above ω is one of the imaginary cube roots of 1.

94. The roots are equal if

$$36\,(2\kappa-1)^2+12\,(\kappa+2)\,(38\kappa+11)=0;$$

$$\therefore\ 50\kappa^2+75\kappa+25=0;$$

$$\therefore\ (\kappa+1)\,(2\kappa+1)=0.$$

Thus the values required are $\kappa=-1$ or $\kappa=-\frac{1}{2}$.

95. We have

$$(ma^3+nc^3)\,(pb^3+qd^3)=(pa^3+qc^3)\,(mb^3+nd^3);$$

$$\therefore\ (mq-np)\,(a^3d^3-b^3d^3)=0.$$

Hence $$mq-np=0, \text{ and } \therefore\ p:q=m:n,$$

or else $$a^3d^3=b^3d^3 \text{ and } \therefore\ ad=bd,$$

(a, b, c, d being all real), so that $a:b=c:d$.

96. Let the population be P_0 at the beginning, P_5 after 5 years, P_{10} after 10 years, and P_{20} after 20 years.

Then $$P_0:P_5=P_5:P_{10};$$

$$\therefore\ P_5{}^2=P_0\times P_{10}.$$

But
$$P_{10}=\frac{108}{100}\cdot P_0;$$
$$\begin{aligned}\therefore\ P_5&=P_0\sqrt{1{\cdot}08}\\&=100000\times 1{\cdot}03923\\&=103923.\end{aligned}$$

Also
$$P_0 : P_{10}=P_{10} : P_{20};$$
$$\begin{aligned}\therefore\ P_0P_{20}&=P_{10}^2=(1{\cdot}08)^2\,.\,P_0^2;\\ \therefore\quad P_{20}&=100000\times(1{\cdot}08)^2\\&=116640.\end{aligned}$$

97. The coefficient of $x^6=-\dfrac{2}{\underline{|6}}-\dfrac{2}{\underline{|2}\,\underline{|4}}+\dfrac{2}{\underline{|5}}+\dfrac{1}{\underline{|3}\,\underline{|3}}$
$$=-\frac{1}{360}-\frac{1}{24}+\frac{1}{60}+\frac{1}{36}=0.$$

98.
$$\frac{1}{2}(a+b)=\frac{5}{4}\sqrt{ab};$$
$$\therefore\ 4(a+b)^2=25ab;$$

i.e.
$$4a^2-17ab+4b^2=0,$$

i.e.
$$(4a-b)(a-4b)=0.$$

Also
$$\sqrt{ab}-\frac{2ab}{a+b}=\frac{4}{5}.$$

Substitute
$$a=4b;$$

then
$$2b-\frac{8b^2}{5b}=\frac{4}{5};\qquad\therefore\ b=2.$$

Thus the numbers are 8 and 2.

99.
$$\begin{aligned}&\frac{1}{b-c}+\frac{1}{c-a}+\frac{1}{a-b}\\&=\frac{(c-a)(a-b)+(a-b)(b-c)+(b-c)(c-a)}{(b-c)(c-a)(a-b)}\\&=\frac{-a^2-b^2-c^2+bc+ca+ab}{(b-c)(c-a)(a-b)}\\&=-\frac{1}{2}\,\frac{(b-c)^2+(c-a)^2+(a-b)^2}{(b-c)(c-a)(a-b)}\\&=\frac{1}{2}\,\frac{(b-c)^2+(c-a)^2+(a-b)^2}{(b-c)(a-c)(a-b)},\end{aligned}$$

which is positive if $a>b>c$.

100.
$$\begin{aligned}\frac{(1-x)^2}{1+x+x^2}&=\frac{(1-x)^3}{1-x^3}\\&=(1-3x+3x^2-x^3)(1+x^3+x^6+\ldots+x^{3r}+\ldots).\end{aligned}$$

The coefficient of $x^{3n} = 1 - 1 = 0$.

.... $x^{3n+2} = 3$.

.... $x^{3n+1} = -3$.

Hence $a_{3n+2} - a_{3n+1} = 3 - (-3) = 6$.

101. $$a(b-c) + b(c-a) + c(a-b) = 0.$$

Hence $$\{a(b-c) + b(c-a) + c(a-b)\}^2 = 0,$$

i.e. $$a^2(b-c)^2 + b^2(c-a)^2 + c^2(a-b)^2$$
$$+ 2bc(c-a)(a-b) + 2ca(a-b)(b-c) + 2ab(b-c)(c-a) = 0.$$

Hence $$\Sigma a^2(b-c)^2 = 2\Sigma bc(a-b)(a-c).$$

102. (i) Arrange in powers of x; then
$$x^2(y+z) + x(y^2 + 2yz + z^2 + 2yz + 2yz - 4yz) + yz(y+z),$$
i.e. $$x^2(y+z) + x(y+z)^2 + yz(y+z),$$
i.e. $$(y+z)\{x^2 + x(y+z) + yz\},$$
i.e. $$(y+z)(z+x)(x+y).$$

(ii) The expression is
$$\{(1+y)^2 - x^2(1-y)^2\}(1-x^2) = \{1 + y - x(1-y)\}\{1 + y + x(1-y)\}(1-x)(1+x).$$

103. (i) $$2x - y = 1,\ 2y - \frac{1}{z} = 1, \text{ and } -7x + \frac{3}{z} = 1;$$
$$\therefore\ 6y - 7x = 4,$$
and $$12x - 6y = 6.$$
$$\therefore\ x = 2 \text{ and } \therefore\ y = 3, \text{ and } z = \frac{1}{5}.$$

(ii) $$ax(ax + 2by) = 3a^2b^2 = 3by(2ax - by);$$
$$\therefore\ a^2x^2 - 4abxy + 3b^2y^2 = 0,$$
i.e. $$(ax - by)(ax - 3by) = 0.$$

If $ax = by$, $3ax^2 = 3ab^2$; $\therefore\ x = \pm b$, and then $y = \pm a$.

If $ax = 3by$, $5by^2 = a^2b$;
$$\therefore\ y = \pm\frac{a}{\sqrt{5}}, \text{ and } \therefore\ x = \pm\frac{3b}{\sqrt{5}}.$$

104. Put $\dfrac{3x^2 + 6x + 1}{4x^2 + 2x - 1} = y$; then
$$(3 - 4y)x^2 + (6 - 2y)x + 1 + y = 0.$$

The roots of the above quadratic will be real, if
$$(3-y)^2 - (3-4y)(1+y) \nless 0,$$
i.e. if $$5y^2 - 5y + 6 \nless 0,$$
i.e. if $$5\left(y - \frac{1}{2}\right)^2 + 6 - \frac{5}{4} \nless 0,$$
and this is the case for *all values* of y, and $\therefore$ real values of x correspond to any proposed value of y.

105. $$(x^2-3xy^{-1}+4y^{-2})^2.$$

Given expression $=(x^2-x+2)^2-6x^2+5x$.

$\therefore$ the given expression is a square if $6x^2-5x=0$,

i.e. if $$x=0, \text{ or } x=\frac{5}{6}.$$

Again, the expression $=(x^2-x-1)^2-x+3$, and is $\therefore$ a square if $x=3$.

106. If the series is a, $a+d$, $a+2d$, ..., we have

$$145=\frac{10}{2}\{2a+9d\},$$

and $$(a+3d)+(a+8d)=5\,(a+2d).$$

Whence $$a=1 \text{ and } d=3.$$

107. $$\frac{1+\sqrt{2}-\sqrt{3}+1+\sqrt{2}+\sqrt{3}}{(1+\sqrt{2})^2-3}+\frac{-1+\sqrt{2}+\sqrt{3}+1-\sqrt{2}+\sqrt{3}}{3-(1-\sqrt{2})^2}$$
$$=\frac{2+2\sqrt{2}}{2\sqrt{2}}+\frac{2\sqrt{3}}{2\sqrt{2}}=1+\frac{1}{\sqrt{2}}+\frac{\sqrt{3}}{\sqrt{2}}$$
$$=\frac{1}{2}\{2+\sqrt{2}+\sqrt{6}\}.$$

108. The families can be represented in $({}_4C_3$, *i.e.* in) 4 ways. And in each case either the husband or the wife can be taken. Hence the whole number is $4\times 2^3=32$.

Or thus: The different committees are shewn by the coefficient of x^3 in the product

$$\{1+(h_1+w_1)\,x\}\,\{1+(h_2+w_2)\,x\}\,\{1+(h_3+w_3)\,x\}\,\{1+(h_4+w_4)\,x\}.$$

Hence the number of ways of choosing the committee is the coefficient of x^3 in $(1+2x)^4$, and this number is

$$\frac{4\,.\,3\,.\,2}{1\,.\,2\,.\,3}\,.\,2^3=32.$$

109. $$(1-2x-3x^2)^{-2}=1+2x\,(2+3x)+3x^2\,(2+3x)^2$$
$$+4x^3\,(2+3x)^3+5x^4\,(2+3x)^4+6x^5\,(2+3x)^5+\dots.$$

Hence the coefficient of x^3 is

$$3\times 12+4\times 2^3=68,$$

and the coefficient of x^5 is

$$4\,.\,3\,.\,2\,.\,3^2+5\,.\,4\,.\,2^3\,.\,3+6\,.\,2^5=888.$$

110. $$(1-x)^{\frac{1}{2}}=1-\frac{1}{2}\,x-\frac{1}{2\,.\,4}\,x^2-\frac{1\,.\,3}{2\,.\,4\,.\,6}\,x^3-\dots\dots$$
$$-\frac{1\,.\,3\,.\,5\,\dots\,(2n-3)}{2\,.\,4\,.\,6\,\dots\,2n}\,x^n-\dots\dots,$$

$$(1-x)^{-1}=1+x+x^2+\dots\dots+x^n+\dots\dots.$$

Hence the coefficient of x^n in $(1-x)^{-\frac{1}{2}}$ is equal to

$$1-\frac{1}{2}-\frac{1}{2.4}-\frac{1.3}{2.4.6}-\dots\dots-\frac{1.3.5\dots(2n-3)}{2.4.6\dots2n}.$$

Hence $$1-S=\frac{1.3.5\dots(2n-1)}{2.4.6\dots2n};$$

$$\therefore\ S=1-\frac{1.3.5\dots(2n-1)}{2.4.6\dots2n}.$$

Or thus:

$$\frac{1.3.5\dots(2n-3)}{2.4.6\dots2n}=\frac{1.3.5\dots(2n-3)}{2.4.6\dots(2n-2)}-\frac{1.3.5\dots(2n-1)}{2.4.6\dots2n},$$

$$\dots\dots=\dots\dots$$

$$\frac{1.3.5}{2.4.6.8}=\frac{1.3.5}{2.4.6}-\frac{1.3.5.7}{2.4.6.8},$$

$$\frac{1.3}{2.4.6}=\frac{1.3}{2.4}-\frac{1.3.5}{2.4.6},$$

$$\frac{1}{2.4}=\frac{1}{2}-\frac{1.3}{2.4},$$

$$\frac{1}{2}=1-\frac{1}{2}.$$

Hence by addition

$$S=1-\frac{1.3.5\dots(2n-1)}{2.4.6\dots2n}.$$

111. $$\begin{aligned}\Sigma a^3(b^2-c^2)&=a^3(b^2-c^2)-a^2(b^3-c^3)+b^2c^2(b-c)\\&=(b-c)\{a^3(b+c)-a^2(b^2+bc+c^2)+b^2c^2\}\\&=(b-c)\{b^2(c^2-a^2)-ba^2(c-a)-ca^2(c-a)\}\\&=(b-c)(c-a)\{b^2(c+a)-a^2(b+c)\}\\&=(b-c)(c-a)\{c(b^2-a^2)+ba(b-a)\}\\&=(b-c)(c-a)(b-a)(bc+ca+ab).\end{aligned}$$

112. $$x-2a=\frac{3ab}{b-a}-2a=\frac{a(b+2a)}{b-a},$$

$$x+b=\frac{3ab}{b-a}+b=\frac{b(b+2a)}{b-a}.$$

Hence $$\begin{aligned}\frac{1}{x-2a}+\frac{2}{x+b}+\frac{1}{b}&=\frac{b-a}{a(b+2a)}+\frac{2(b-a)}{b(b+2a)}+\frac{1}{b}\\&=\frac{(b-a)(b+2a)}{ab(b+2a)}+\frac{1}{b}\\&=\frac{b-a}{ab}+\frac{1}{b}=\frac{1}{a}.\end{aligned}$$

113. (i) Multiply by $(x-a)(x-b)$; then

$$b(x-b)+a(x-a)=2(x-a)(x-b);$$

$$\therefore\ 2x^2-3(a+b)x+(a+b)^2=0,$$

i.e. $$\{2x-(a+b)\}\{x-(a+b)\}=0.$$

(ii) Put $$\frac{x+2}{x^2-1}=y;$$

then $$y+\frac{1}{y}=\frac{25}{12};$$

$$\therefore\ 12y^2-25y+12=0,$$

i.e. $$(4y-3)(3y-4)=0.$$

Hence $$(\alpha)\quad \frac{x+2}{x^2-1}=\frac{4}{3},$$

and $$\therefore\ 4x^2-3x-10=0,$$

whence $$x=2 \text{ or } x=-\frac{5}{4};$$

or $$(\beta)\quad \frac{x+2}{x^2-1}=\frac{3}{4},$$

and $$\therefore\ 3x^2-4x-11=0,$$

whence $$x=\frac{1}{3}(2\pm\sqrt{37}).$$

(iii) $$x(x+1)=6y, \text{ and } x^3+1=9y.$$

By division $$\frac{x(x+1)}{(x+1)(x^2-x+1)}=\frac{6}{9}=\frac{2}{3}.$$

Hence $$x+1=0 \text{ and then } y=0,$$

or $$3x=2(x^2-x+1);$$

$$\therefore\ 2x^2-5x+2=0,$$

i.e. $$(2x-1)(x-2)=0.$$

When $$x=2,\quad y=1,$$

and when $$x=\frac{1}{2},\quad y=\frac{1}{8}.$$

Thus the roots are

$$-1,\ 0;\ 2,\ 1;\ \frac{1}{2},\ \frac{1}{8}.$$

114. Let A do the whole work in x days and B in y days. Then between them they do $\left(\frac{1}{x}+\frac{1}{y}\right)$th per day, and they will ∴ do the whole in $\frac{xy}{x+y}$ days.

Hence $$\frac{xy}{x+y}=\frac{x}{2}+1=\frac{y}{2}-2.$$

Hence $$x=y-6,$$

and $$\frac{y(y-6)}{2y-6}=\frac{y}{2}-2;$$

$$\therefore\ 2y(y-6)=(y-4)(2y-6);$$

$$\therefore\ y=12 \text{ and } x=6.$$

Hence the time required $=\dfrac{12\times 6}{12+6}=4$ days.

115. $$\frac{x+2a}{x-2a}+\frac{x+2b}{x-2b}=2,$$

if $$x^2+2(a-b)x-4ab+x^2-2(a-b)x-4ab=2\{x^2-2x(a+b)+4ab\},$$

i.e. if $$4x(a+b)=16ab,$$

i.e. if $$x=4ab/(a+b).$$

116. The expression arranged in powers of x is

$$\begin{aligned}&x^4+2x^3(y+z)+x^2(y^2+z^2+4yz)+2xyz(y+z)+y^2z^2\\&=\{x^2+x(y+z)+yz\}^2\\&=(x+y)^2(x+z)^2.\end{aligned}$$

117. $$\begin{aligned}S_{2n+1}&=(n^2+1)+(n^2+2)+\ldots \text{ to } (2n+1) \text{ terms}\\&=\frac{2n+1}{2}\{2n^2+2+2n\}\\&=(n^2+n+1)(2n+1)\\&=2n^3+3n^2+3n+1=n^3+(n+1)^3.\end{aligned}$$

118. Eliminate z; then

$$c^2(a-b)^2xy-\{a(b-c)y+b(c-a)x\}\{a(b-c)x+b(c-a)y\}=0;$$

$$\therefore\ -ab(b-c)(c-a)(x^2+y^2)+\{c^2(a-b)^2-a^2(b-c)^2-b^2(c-a)^2\}xy=0;$$

$$\therefore\ -ab(b-c)(c-a)(x^2-2xy+y^2)=0,$$

for $$c^2(a-b)^2-a^2(b-c)^2-b^2(c-a)^2=-2ab(c-a)(c-b).$$

Hence $x=y$, for $ab(b-c)(c-a)$ is not zero. Similarly $x=z$.

119. $$(1-2x-4x^2)^{-3}=1+3x(2+4x)+\frac{3\cdot 4}{1\cdot 2}x^2(2+4x)^2$$

$$+\frac{4\cdot 5}{1\cdot 2}x^3(2+4x)^3+\frac{5\cdot 6}{1\cdot 2}x^4(2+4x)^4+\ldots.$$

Hence the coefficient of x^3

$$=\frac{4\cdot 5}{1\cdot 2}\cdot 8+\frac{3\cdot 4}{1\cdot 2}\cdot 16=176.$$

Also the coefficient of x^4

$$\begin{aligned}&=\frac{5\cdot 6}{1\cdot 2}\cdot 2^4+\frac{4\cdot 5}{1\cdot 2}\cdot 3\cdot 2^2\cdot 4+\frac{3\cdot 4}{1\cdot 2}\cdot 16\\&=240+480+96=816.\end{aligned}$$

120.
$$(1-x)^{\frac{1}{2}}=1-\frac{1}{2}x-\frac{1.1}{2.4}x^2-\frac{1.1.3}{2.4.6}x^3-\ldots,$$
$$(1-x)^{-1}=1+x+x^2+x^3+\ldots.$$

Hence the coefficient of x^n in $(1-x)^{\frac{1}{2}}\times(1-x)^{-\frac{1}{2}}$ is
$$1-\frac{1}{2}-\frac{1}{2.4}-\frac{1.3}{2.4.6}-\ldots \text{ to } n \text{ terms.}$$

Hence the sum of $\frac{1}{2.4}+\frac{1.3}{2.4.6}+\ldots$ to infinity is equal to $\frac{1}{2}$ if the coefficient of the nth term of the expansion of $(1-x)^{-\frac{1}{2}}$ vanishes when n is very great.

Thus we have to prove that
$$\frac{1.3.5\ldots(2n-1)}{2.4.6\ldots2n}$$
vanishes when n is very great.

Now
$$\frac{1}{2}<\frac{2}{3},\ \frac{3}{4}<\frac{4}{5},\ \ldots\ \frac{2n-1}{2n}<\frac{2n}{2n+1}.$$

Hence
$$\frac{1.3.5\ldots(2n-1)}{2.4.6\ldots2n}<\frac{2.4.6\ldots2n}{3.5.7\ldots2n+1};$$
$$\therefore\ \left\{\frac{1.3.5\ldots(2n-1)}{2.4.6\ldots2n}\right\}^2<\frac{1}{2n+1}.$$

But $\frac{1}{2n+1}=0$ when n is ∞ and $\therefore\ \frac{1.3.5\ldots(2n-1)}{2.4.6\ldots2n}=0$ when n is ∞.

121. If
$$\frac{x}{b-c}=\frac{y}{c-a}=\frac{z}{a-b}=t;$$
then
$$x(b+c)=t(b-c)(b+c),$$
$$y(c+a)=t(c-a)(c+a),$$
$$z(a+b)=t(a-b)(a+b),$$
$$\therefore\ \Sigma x(b+c)=t(b^2-c^2+c^2-a^2+a^2-b^2)=0.$$

122. Divide in the usual manner, and the remainder will be found to be
$$-a^6-\frac{1}{27}p^6=-\frac{1}{27}(p^6+27a^6).$$

Hence the required condition is that $p^6+27a^6=0$.

123. We have to prove that
$$(1-a)x+(1-b)y+(1-c)z=0,$$
and that
$$(1-a^2)x+(1-b^2)y+(1-c^2)z=0.$$

Now $\Sigma(1-a)x$

$$=(1-a)(1-b)(1-c)\left\{\frac{1}{(a-b)(a-c)}+\frac{1}{(b-c)(b-a)}+\frac{1}{(c-a)(c-b)}\right\}$$

$$=\frac{(1-a)(1-b)(1-c)}{(b-c)(c-a)(a-b)}\{c-b+a-c+b-a\}$$

$$=0.$$

Also $\Sigma(1-a^2)x$

$$=(1-a)(1-b)(1-c)\left\{\frac{1+a}{(a-b)(a-c)}+\frac{1+b}{(b-c)(b-a)}+\frac{1+c}{(c-a)(c-b)}\right\}$$

$$=\frac{(1-a)(1-b)(1-c)}{(b-c)(c-a)(a-b)}\{(1+a)(c-b)+(1+b)(a-c)+(1+c)(b-a)\}$$

$$=0.$$

124. We have

$$x^2+x+1=3\sqrt{\{3(x^2+x+1)-14\}},$$

$$\therefore\ (x^2+x+1)^2=27(x^2+x+1)-126,$$

i.e.
$$(x^2+x+1-6)(x^2+x+1-21)=0,$$

whence the roots are

$$4,\ -5,\ \frac{1}{2}(-1\pm\sqrt{21}).$$

125. Let the trains travel at x and y miles per hour respectively, then they approach one another at $x+y$ miles per hour, and we have

$$\frac{15}{x+y}=\frac{15}{x}-\frac{18}{60}=\frac{15}{y}-\frac{8}{60}.$$

Hence
$$\frac{15y}{x(x+y)}=\frac{18}{60},\ \text{and}\ \frac{15x}{y(x+y)}=\frac{8}{60}.$$

Hence
$$\frac{15^2}{(x+y)^2}=\frac{18}{60}\times\frac{8}{60}=\frac{1}{25};$$

$$\therefore\ x+y=15\times5=75.$$

Then
$$\frac{15}{75}=\frac{15}{x}-\frac{18}{60},$$

whence $x=30$ and $\therefore\ y=45$.

126. Put
$$\frac{6x^2-22x+21}{5x^2-18x+17}=y;$$

then
$$x^2(6-5y)-2x(11-9y)+21-17y=0.$$

Since x is real,

$$(11-9y)^2-(6-5y)(21-17y)\not<0,$$

i.e.
$$-4y^2+9y-5\not<0,$$

i.e.
$$-(4y-5)(y-1)\not<0.$$

Hence y must lie between 1 and $\frac{5}{4}$.

Also when $$y=1, \quad x=2;$$

and when $$y=\frac{5}{4}, \quad x=1.$$

127. $$\frac{(4+\sqrt{15})^{\frac{3}{2}}+(4-\sqrt{15})^{\frac{3}{2}}}{(6+\sqrt{35})^{\frac{3}{2}}-(6-\sqrt{35})^{\frac{3}{2}}}=\frac{(8+2\sqrt{15})^{\frac{3}{2}}+(8-2\sqrt{15})^{\frac{3}{2}}}{(12+2\sqrt{35})^{\frac{3}{2}}-(12-2\sqrt{35})^{\frac{3}{2}}}$$
$$=\frac{(\sqrt{5}+\sqrt{3})^3+(\sqrt{5}-\sqrt{3})^3}{(\sqrt{7}+\sqrt{5})^3-(\sqrt{7}-\sqrt{5})^3}=\frac{2\,(5\sqrt{5}+3\sqrt{5}\,.\,3)}{2\,(3\,.\,7\sqrt{5}+5\sqrt{5})}=\frac{7}{13}.$$

128. $$S_1=1+2+3+\ldots\ldots \text{ to } n \text{ terms.}$$
$$S_2=2+5+8+\ldots\ldots \quad \ldots\ldots\ldots\ldots$$
$$\ldots=\ldots\ldots\ldots$$
$$S_m=m+(m+\overline{2m-1})+\ldots\ldots \text{ to } n \text{ terms.}$$

Hence $$S_1=\frac{n}{2}\{2+(n-1)\}, \qquad S_2=\frac{n}{2}\{4+(n-1)\,3\}, \ \ldots\ldots,$$
$$S_m=\frac{n}{2}\{2m+\overline{n-1}\ \overline{2m-1}\};$$
$$\therefore \ S_1+S_2+\ldots\ldots+S_m=n\,(1+2+\ldots\ldots+m)$$
$$+\frac{1}{2}n\,(n-1)\{1+3+5+\ldots\ldots+(2m-1)\}$$
$$=\frac{1}{2}\,nm\,(m+1)+\frac{1}{2}\,n\,(n-1)\,.\,m^2$$
$$=\frac{1}{2}\,mn\,(m+1+mn-m)$$
$$=\frac{1}{2}\,mn\,(mn+1).$$

129. The expansion is
$$a^n+na^{n-1}b+\frac{n\,(n-1)}{1\,.\,2}\,a^{n-2}b^2+\ldots.$$

Hence $$a^n=1, \quad na^{n-1}b=1,$$

and $$\frac{n\,(n-1)}{1\,.\,2}\,a^{n-2}b^2=\frac{2}{3}.$$

Hence $a=1$, then $nb=1$, and $\frac{nb\,(nb-b)}{1\,.\,2}=\frac{2}{3}$,

whence $$b=-\frac{1}{3} \text{ and } n=-3.$$

Thus the expansion is that of $\left(1-\frac{1}{3}\right)^{-3}$.

130. The coefficient of x^{2n} in $(1+x)^{4n}$ is

$$\frac{4n\,!}{2n\,!\;2n\,!}.$$

Also

$$\{1+x\,(2+x)\}^{2n}=1+2n\,.\,x\,(2+x)+\dots$$
$$+\frac{(2n)!}{n!\;n!}\,x^n\,(2+x)^n+\frac{(2n)!}{(n+1)!\;(n-1)!}\,x^{n+1}\,(2+x)^{n+1}$$
$$+\frac{(2n)!}{(n+2)!\;(n-2)!}\,x^{n+2}\,(2+x)^{n+2}+\dots+x^{2n}\,(2+x)^{2n}.$$

The coefficient of x^{2n} in the last expression

$$=\frac{(2n)!}{n!\;n!}+\frac{(2n)!}{(n+1)!\;(n-1)!}\cdot\frac{(n+1)!}{(n-1)!\;2!}\,2^2$$
$$+\frac{(2n)!}{(n+2)!\;(n-2)!}\cdot\frac{(n+2)!}{(n-2)!\;4!}\,2^4+\dots$$
$$+\frac{(2n)!}{(n+r)!\;(n-r)!}\cdot\frac{(n+r)!}{(n-r)!\;(2r)!}\,2^{2r}+\dots+2^{2n}.$$

Hence

$$\frac{(4n)!}{(2n)!\;(2n)!}=\frac{(2n)!}{n!\;n!}+\frac{(2n)!}{(n-1)!\;(n-1)!\;2!}\,2^2+\dots$$
$$+\frac{(2n)!}{(n-r)!\;(n-r)!\;2r!}\,2^{2r}+\dots+2^{2n}.$$

131.

$$\left(\frac{1}{y-z}+\frac{1}{z-x}+\frac{1}{x-y}\right)^2=\frac{1}{(y-z)^2}+\frac{1}{(z-x)^2}+\frac{1}{(x-y)^2}$$
$$+\frac{2}{(z-x)\,(x-y)}+\frac{2}{(x-y)\,(y-z)}+\frac{2}{(y-z)\,(z-x)}.$$

But

$$\Sigma\,\frac{1}{(z-x)\,(x-y)}=\frac{(y-z)+(z-x)+(x-y)}{(y-z)\,(z-x)\,(x-y)}=0.$$

132. Any common factor of the given expressions A and B is also a factor of

$$A-xyB,\text{ that is of }(y^2+y)\,x+5y-1.$$

Also any common factor of B and $(y^2+y)\,x+5y-1$ is a factor of

$$2\,(y^2+y)\,x^2+(11y-2)\,x+4-2x\,[(y^2+y)\,x+5y-1],$$

that is of $xy+4$.

Hence if A and B have a common factor which contains x, the expressions

$$(y^2+y)\,x+5y-1\text{ and }(y+1)\,\{xy+4\}$$

must be identical. Hence $5y-1=4y+4$, so that $y=5$ and when $y=5$ the common factor is $5x+4$ or a numerical multiple of $5x+4$.

133.

$$(xy-1)^2>(x^2-1)\,(y^2-1),$$

if

$$x^2y^2-2xy+1>x^2y^2-x^2-y^2+1,$$

if

$$x^2-2xy+y^2>0,$$

and $(x-y)^2$ is >0 for all real values of x and y.

134. (i) $$\frac{1}{x}+\frac{1}{y}=\frac{x+y}{6};$$

$$\therefore\ 6(x+y)=xy(x+y).$$

Hence $xy=6$, for $x+y$ cannot be zero.

Also $$(x+y)^2+(x+y)-30=0,$$

i.e. $$(x+y+6)(x+y-5)=0.$$

From $$x+y=5 \text{ and } xy=6,$$

we have the solutions $$2,\ 3;\ \ 3,\ 2.$$

From $$x+y=-6 \text{ and } xy=6,$$

we have $$x=-3\pm\sqrt{3} \text{ and } y=-3\mp\sqrt{3}.$$

(ii) We have $$(y+1)(z+1)=18,$$
$$(z+1)(x+1)=9,$$
$$(x+1)(y+1)=8;$$
$$\therefore\ (x+1)^2=9\times 8\div 18=4.$$

Hence $$x+1=\pm 2;$$
$$\therefore\ \ y+1=\pm 4,$$

and $$z+1=\pm\frac{9}{2}.$$

Hence $1,\ 3,\ 3\frac{1}{2};\ -3,\ -5,\ -5\frac{1}{2}$ are the solutions.

135. $$\begin{array}{ccc}
\pounds & s. & d.\\
x & y & z\\
2 & 19 & 9\\
\hline
z & y & x
\end{array}
\qquad
\begin{array}{ccc}
\pounds & s. & d.\\
x & y & z\\
 & 2 & 9\\
\hline
x & z & y
\end{array}$$

It is obvious that $z>x$.

Hence, from the first condition, we have
$$9+z=x+12;\quad \therefore\ z=x+3.$$

And, from the second condition, we have $y>z$, and
$$z+12-9=y;\quad \therefore\ y=z+3=x+6.$$

If now £5. 14*s*. be added, we have
$$y+14=x+6+14=x+20,$$

and then $$x+5+1=x+6=y.$$

Hence x and y are interchanged.

136. From $$x+\frac{a^2}{x}=y+\frac{a^2}{y},$$

we have $$x-y+a^2\left(\frac{1}{x}-\frac{1}{y}\right)=0,\quad \textit{i.e. } (x-y)\left(1-\frac{a^2}{xy}\right)=0.$$

Similarly $$(x-z)\left(1-\frac{a^2}{xz}\right)=0.$$

Hence (i) $x=y$ and $x=z$,

or (ii) $x=y$ and $1-\dfrac{a^2}{xz}=0$,

or (iii) $x=z$ and $1-\dfrac{a^2}{xy}=0$,

or (iv) $1-\dfrac{a^2}{xy}=0$ and $1-\dfrac{a^2}{xz}=0$.

From (i) $x=y=z=$ anything.

From (ii) $x=y=a^2/z=$ anything.

From (iii) $x=z=a^2/y=$ anything.

From (iv) $y=z=a^2/x=$ anything.

137.
$$\frac{1}{\sqrt{2}}=\frac{1}{2}\sqrt{2},\quad \frac{1}{\sqrt{2}+1}=\sqrt{2}-1,$$

and
$$\frac{1}{3\sqrt{2}+4}=\frac{3\sqrt{2}-4}{18-16}=\frac{3}{2}\sqrt{2}-2.$$

Hence the three are in A.P. whose common difference is $\frac{1}{2}\sqrt{2}-1$.

138. If a, b, c, d are in A.P.,
$$a-d=3\,(a-b)=3\,(c-d);$$
$$\therefore\ 3b=2a+d \text{ and } 3c=a+2d.$$

And, if a, g, h, d are in H.P.
$$\frac{1}{a}-\frac{1}{d}=3\left(\frac{1}{a}-\frac{1}{g}\right)=3\left(\frac{1}{h}-\frac{1}{d}\right);$$
$$\therefore\ g=\frac{3ad}{a+2d} \text{ and } h=\frac{3ad}{2a+d}.$$

Hence $bh=ad$ and $gc=ad$.

Also, if a, e, f, d are in G.P.,
$$\frac{a}{e}=\frac{f}{d} \text{ and } \therefore\ ad=ef.$$

139. Substitute $\dfrac{2}{2-x}$ for x; then we have
$$\frac{\{16/(2-x)^4-32/(2-x)^3+16/(2-x)-4\}^2}{\{4/(2-x)^2-4/(2-x)+2\}^4}$$
$$=\frac{\{4-8\,(2-x)+4\,(2-x)^3-(2-x)^4\}^2}{\{2-2\,(2-x)+(2-x)^2\}^4}$$
$$=\frac{(4-8x+4x^3-x^4)^2}{(2-2x+x^2)^4}.$$

140.
$$\{(1-x-y-z)(1-x-y-z+4xyz)\}^{\frac{1}{2}}$$
$$=(1-x-y-z)\left(1+\frac{4xyz}{1-x-y-z}\right)^{\frac{1}{2}}$$
$$=(1-x-y-z)\left\{1+\frac{1}{2}\cdot\frac{4xyz}{1-x-y-z}-\frac{1}{2\,.\,4}\cdot\frac{16x^2y^2z^2}{(1-x-y-z)^2}-\dots\right\}.$$

Hence the coefficient of xyz is 2.

141. Arrange the dividend in ascending powers of a; then

$$\begin{array}{l}
x+y-4a\,)\,x^3+y^3-6a\,(x^2+y^2)+11a^2\,(x+y)-12a^3\,(\,x^2-xy+y^2-2a\,(x+y)+3a^2 \\
\qquad\qquad \underline{x^3+y^3-4a\,(x^2-xy+y^2)} \\
\qquad\qquad\qquad\qquad -2a\,(x+y)^2+11a^2\,(x+y) \\
\qquad\qquad\qquad\qquad \underline{-2a\,(x+y)^2+\ \ 8a^2\,(x+y)} \\
\qquad\qquad\qquad\qquad\qquad\qquad 3a^2\,(x+y)-12a^3 \\
\qquad\qquad\qquad\qquad\qquad\qquad \underline{3a^2\,(x+y)-12a^3}
\end{array}$$

142. $(cy+bz)^3+(ay+cz)^3+(by+az)^3-3\,(cy+bz)\,(ay+cz)\,(by+az)$
$$=\{(cy+bz)+(ay+cz)+(by+az)\}\,\{\Sigma\,(cy+bz)^2-\Sigma\,(cy+bz)\,(ay+cz)\}.$$

Now $\quad (cy+bz)+(ay+cz)+(by+az)=(a+b+c)\,(y+z)=0,$

since $\quad a+b+c=0.$

Hence the sum of the cubes of $cy+bz$, $ay+cz$ and $(by+az)$ minus thrice their product is zero.

143. (i) Multiply by $\sqrt{(x-3)}$; then
$$(x-3)\sqrt{(x-2)}+5\sqrt{(x-2)}=\sqrt{\{(x+2)\,(x+4)\,(x-3)\}}.$$

Square both sides; then
$$(x-3)^2\,(x-2)+25\,(x-2)+10\,(x-3)\,(x-2)=(x+2)\,(x+4)\,(x-3);$$
$$\therefore\ x^3-8x^2+21x-18+25x-50+10x^2-50x+60=x^3+3x^2-10x-24;$$
$$\therefore\ x^2-6x-16=0,\quad i.e.\ (x-8)\,(x+2)=0.$$

(ii) $$ax-by=a^2+b^2,$$

and $$(a^2-b^2)\,(bx+ay)=(a^2+b^2)\,xy.$$

Substitute for y; then
$$(a^2-b^2)\,\{b^2x+a^2x-a\,(a^2+b^2)\}=(a^2+b^2)\,x\,(ax-a^2-b^2);$$
$$\therefore\ (a^2-b^2)\,\{x-a\}=x\,(ax-a^2-b^2);$$
$$\therefore\ x^2-2ax+(a^2-b^2)=0;$$
$$\therefore\ x=a\pm b.$$

And then $$y=(ax-a^2-b^2)/b=\pm a-b.$$

(iii) $$x^3+1=81\,(y^2+y),$$

and $$x^2+x=\ 9\,(y^3+1);$$
$$\therefore\ x^3+3x^2+3x+1=27\,(y^3+3y^2+3y+1),$$

i.e. $$(x+1)^3=27\,(y+1)^3;$$
$$\therefore\ x+1=3\,(y+1),\ \text{or}\ x+1=3\omega\,(y+1),\ \text{or}\ x+1=3\omega^2\,(y+1).$$

If $$x+1=3(y+1),$$

we have $$(3y+2)^3+1=81(y^2+y);$$

$$\therefore\ 3y^3-3y^2-5y+1=0,$$

i.e. $$(y+1)(3y^2-6y+1)=0;$$

$$\therefore\ y=-1 \text{ or } y=1\pm\frac{1}{3}\sqrt{6}.$$

Then $$x=-1 \text{ or } x=5\pm\sqrt{6}.$$

The imaginary roots can be found in a similar manner from

$$x+1=3\omega(y+1).$$

144. Let there be x gallons of wine and y of water.

Then $$\frac{x}{y}=\frac{5}{3}.$$

Now 16 gallons of the mixture contains 10 gallons of wine and 6 of water. Hence after the substitution of the water, there will be $x-10$ gallons of wine and $y+10$ of water, so that

$$\frac{x-10}{y+10}=\frac{3}{5}.$$

Hence $$3x=5y,$$

and $$5x-50=3y+30,$$

whence $$x=25 \text{ and } y=15.$$

145. Each fraction

$$=\frac{(bx-ay)+(cx-az)+a(z+y)}{(cy-az)+(by-ax)+a(x+z)}$$

$$=\frac{(b+c)x}{(b+c)y}=\frac{x}{y}, \text{ unless } b+c=0.$$

146. Let a, ar, ar^2 be the three quantities in G.P.; then we have to prove that

$$a+ar, \quad 2ar, \quad ar+ar^2$$

are in H.P., or that

$$\frac{1}{a(1+r)}, \ \frac{1}{2ar} \text{ and } \frac{1}{ar(1+r)}$$

are in A.P.; or that

$$\frac{2}{2ar}=\frac{1}{a(1+r)}+\frac{1}{ar(1+r)},$$

which is obvious.

147. We have $2b=a+c$ and $b^2=a(c+1)$.

Hence $$b^2-a=ac=2ab-a^2; \quad \therefore\ a=(a-b)^2=(b-c)^2.$$

148. If x and y are positive and

$$x<\frac{1-y}{1+y},$$

it follows that $y<1$ and $x<1$.

Then from

$$x+xy<1-y,$$

$$xy+y<1-x;$$

$$\therefore\ y<\frac{1-x}{1+x}.$$

149.

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=\left(\frac{x}{a}+\frac{y}{b}\right)^2-2\frac{xy}{ab};$$

$$\therefore\ 2=(x/a+y/b)^2+2, \text{ and } \therefore\ x/a+y/b=0.$$

Then

$$\frac{b}{x}-\frac{a^2}{bx}=1;\quad \therefore\ x=-\frac{a^2-b^2}{b} \text{ and } \therefore\ y=\frac{a^2-b^2}{a}.$$

Hence

$$-(a^2-b^2)^2/ab=xy=-ab;$$

$$\therefore\ (a^2-b^2)^2=a^2b^2.$$

150.

$$(1-x+x^2)^{-1}=(1+x)(1+x^3)^{-1}$$

$$=(1+x)(1-x^3+x^6-\ldots+(-1)^r x^{3r}+\ldots).$$

Hence coefficient of x^{3n} is $(-1)^n=p_{3n}$.

Also

$$\frac{1}{1-x-2x^2}=\frac{2}{3}(1-2x)^{-1}+\frac{1}{3}(1+x)^{-1}$$

$$=\frac{2}{3}\{1+2x+(2x)^2+\ldots+(2x)^r+\ldots\}$$

$$+\frac{1}{3}\{1-x+x^2+\ldots+(-1)^r x^r+\ldots\}.$$

Hence

$$q_n=\frac{2}{3}\cdot 2^n+\frac{1}{3}(-1)^n.$$

$$\therefore\ 3q_n=2^{n+1}+(-1)^n=2^{n+1}+p_{3n}.$$

151.

$$4\Sigma(\sigma^3-a^3)(s-a)=(-a^3+b^3+c^3)(-a+b+c)$$

$$+(a^3-b^3+c^3)(a-b+c)+(a^3+b^3-c^3)(a+b-c)$$

$$=\{3\Sigma a^4-\Sigma b^3c\}.$$

Also

$$\Sigma a^4-s\sigma^3=\Sigma a^4-\frac{1}{4}(a^3+b^3+c^3)(a+b+c)$$

$$=\frac{1}{4}\{4\Sigma a^4-\Sigma a^4-\Sigma b^3c\}.$$

Hence

$$\Sigma(\sigma^3-a^3)(s-a)=\Sigma a^4-s\sigma^3.$$

152. $(bz-cy)(a+x)+(cx-az)(b+y)+(ay-bx)(c+z)=0.$

Divide the terms in order by a^2-x^2, b^2-y^2, c^2-z^2 which are all equal; then

$$\frac{(bz-cy)(a+x)}{a^2-x^2}+\frac{(cx-az)(b+y)}{b^2-y^2}+\frac{(ay-bx)(c+z)}{c^2-z^2}=0;$$

$$\therefore\ \Sigma\frac{bz-cy}{a-x}=0.$$

153. Since $x+y+z=0,\quad \Sigma x^2+2\Sigma yz=0;$

and, since $a+b+c=0,\quad \Sigma a^2+2\Sigma bc=0.$

Now
$$\begin{aligned}\Sigma(x^2+a^2)(yz-bc)&\\ &=\Sigma x^2yz-\Sigma a^2bc+\Sigma a^2yz-\Sigma x^2bc\\ &=xyz\Sigma x-abc\Sigma a+\Sigma a^2yz-\Sigma x^2bc\\ &=\Sigma a^2yz-\Sigma x^2bc\\ &=\Sigma a^2(z+x)(x+y)-\Sigma x^2(c+a)(a+b)\\ &=\Sigma a^2x^2+\Sigma a^2\,.\,\Sigma yz-\Sigma x^2a^2-\Sigma x^2\,.\,\Sigma bc\\ &=-2\Sigma bc\,.\,\Sigma yz+2\Sigma yz\,.\,\Sigma bc=0.\end{aligned}$$

154. (i)
$$x(y+z)=14,$$
$$y(z+x)=18,$$
$$z(x+y)=20.$$

Subtract each from the sum of the other two; then

$$2yz=24,\quad 2zx=16,\quad 2xy=12.$$

Hence
$$x^2y^2z^2=12\times8\times6;$$
$$\therefore\ xyz=\pm24;$$
$$\therefore\ \frac{x}{2}=\frac{y}{3}=\frac{z}{4}=\pm1.$$

(ii)
$$y+z=6xyz,$$
$$z+x=10xyz,$$
$$x+y=12xyz.$$

Subtract each from the sum of the other two; then we have

$$x=8xyz,\quad y=4xyz,\quad z=2xyz.$$

Hence
$$xyz=8\,.\,4\,.\,2\,(xyz)^3;$$

$\therefore\ xyz=0$ from which we get $x=y=z=0$,

or
$$xyz=\pm\frac{1}{8}.$$

Hence
$$x=2y=4z=\pm1.$$

155. Let A travel x miles per day; then B travels $x-6$ miles per day. They approach at $2x-6$ miles per day, and $\therefore$ meet in $180/(2x-6)$ days.

Hence $\qquad x-6=\dfrac{360}{2x-6},$

whence $\qquad x^2-9x-162=0,$

i.e. $\qquad (x-18)(x+9)=0.$

Thus A goes 18 miles and B goes 12 miles a day. [The negative value of x is meaningless.]

156. $\qquad y-z+x : z-x+y=x-y+z : x+y+z,$

if $\qquad (y-z+x)(x+y+z)=(z-x+y)(x-y+z),$

i.e if $\qquad x^2+y^2-z^2+2xy=z^2-x^2-y^2+2xy,$

i.e. if $\qquad x^2+y^2=z^2.$

157. Let a and b be the given quantities, then the inserted means will be $\frac{1}{3}(2a+b)$ and $\frac{1}{3}(a+2b)$.

Now $\qquad \frac{1}{3}(2a+b)\cdot\frac{1}{3}(a+2b)>ab,$

if $\qquad (2a+b)(a+2b)-9ab>0,$

i.e. if $\qquad 2a^2-4ab+2b^2>0,$

i.e. if $\qquad 2(a-b)^2>0,$

which is obvious.

158. If a, b, c are in H.P., $b=2ac/(a+c)$.

Hence $\qquad a+c>2b,$

if $\qquad a+c>\dfrac{4ac}{a+c},$

i.e. if $\qquad (a+c)^2>4ac,$

since a and c are both positive;

i.e. if $\qquad (a-c)^2>0.$

Again, since $\qquad (a+c)^2>2^2ac,$

$$(a+c)^n>2^n a^{\frac{n}{2}} c^{\frac{n}{2}},$$

if n be positive, and $\qquad a^n+c^n>2a^{\frac{n}{2}}c^{\frac{n}{2}}.$

Hence $\qquad (a^n+c^n)(a+c)^n>2^{n+1}a^n c^n,$

$$\therefore\ a^n+c^n>2\cdot\left(\frac{2ac}{a+c}\right)^n>2b^n.$$

159. Put $\qquad ax^2=by^2=cz^2=\lambda^2;$

then $\qquad x=\lambda/\sqrt{a},\ y=\lambda/\sqrt{b}$ and $z=\lambda/\sqrt{c}.$

Hence $\qquad a^2\cdot\dfrac{\lambda^3}{a\sqrt{a}}+b^2\cdot\dfrac{\lambda^3}{b\sqrt{b}}+c^2\cdot\dfrac{\lambda^3}{c\sqrt{c}}=p^5;$

$$\therefore\ \lambda^3=p^5/(\sqrt{a}+\sqrt{b}+\sqrt{c}).$$

Also $$\sqrt{a}/\lambda+\sqrt{b}/\lambda+\sqrt{c}/\lambda=1/p\,;$$
$$\therefore\ \lambda^3=p^3(\sqrt{a}+\sqrt{b}+\sqrt{c})^3.$$

Hence $$p^2=(\sqrt{a}+\sqrt{b}+\sqrt{c})^4,$$

or $$\sqrt{p}=\sqrt{a}+\sqrt{b}+\sqrt{c}.$$

160. $$\left(1-\frac{1}{2}\right)^{-2n}=1+\frac{2n}{1}\cdot\frac{1}{2}+\frac{2n(2n+1)}{1.2}\left(\frac{1}{2}\right)^2+\frac{2n(2n+1)(2n+2)}{1.2.3}\left(\frac{1}{2}\right)^3+\dots$$
$$=1+n+\frac{n\left(n+\frac{1}{2}\right)}{1.2}+\frac{n\left(n+\frac{1}{2}\right)(n+1)}{1.2.3}+\dots,$$
$$\left(1+\frac{1}{2}\right)^{2n}=1+\frac{2n}{1}\cdot\frac{1}{2}+\frac{2n(2n-1)}{1.2}\left(\frac{1}{2}\right)^2+\frac{2n(2n-1)(2n-2)}{1.2.3}\cdot\left(\frac{1}{2}\right)^3+\dots$$
$$=1+n+\frac{n\left(n-\frac{1}{2}\right)}{1.2}+\frac{n\left(n-\frac{1}{2}\right)(n-1)}{1.2.3}+\dots.$$

Hence the product of the two series
$$=\left(1-\frac{1}{2}\right)^{-2n}\times\left(1+\frac{1}{2}\right)^{2n}=2^{2n}\times\left(\frac{3}{2}\right)^{2n}=3^{2n}.$$

Cambridge:
PRINTED BY JOHN CLAY, M.A.
AT THE UNIVERSITY PRESS.

WORKS BY THE REV. J. B. LOCK, M.A.

ARITHMETIC FOR SCHOOLS. New Edition, revised and enlarged. With or without Answers. Globe 8vo. 4*s.* 6*d.* Also issued in Two Parts. Part I. 2*s.* Part II. 3*s.*

KEY TO ARITHMETIC FOR SCHOOLS. By the Rev. R. G. Watson, M.A., formerly Head Master of the Grammar School, Halstead, Vicar of Gussage, All Saints, Dorset. Crown 8vo. 10*s.* 6*d.*

EXAMPLES IN ARITHMETIC FOR SCHOOLS. Globe 8vo. 3*s.* Answers. Globe 8vo. Sewed. 6*d.*

ARITHMETIC FOR SCHOOLS. Revised and enlarged with the assistance of V. M. Turnbull, M.A. Part I. With Answers, 2*s.* Part II. With Answers, 3*s.* Complete with or without Answers. Extra crown 8vo. 4*s.* 6*d.*

ARITHMETIC FOR BEGINNERS. A School Class-Book of Commercial Arithmetic. Globe 8vo. 2*s.* 6*d.*

KEY TO ARITHMETIC FOR BEGINNERS. By the Rev. R. G. Watson, M.A. Crown 8vo. 8*s.* 6*d.*

A SHILLING BOOK OF ARITHMETIC. Pott 8vo. 1*s.* Also with Answers. 1*s.* 6*d.*

KEY TO SHILLING ARITHMETIC. By Henry Carr, B.A., of Her Majesty's Education Office, Lagos, West Africa. Crown 8vo. 8*s.* 6*d.* net.

A TREATISE ON ELEMENTARY TRIGONOMETRY. Extra Fcap. 8vo. 4*s.* 6*d.*

KEY TO ELEMENTARY TRIGONOMETRY. By Henry Carr, B.A., of the Grammar School, Lagos, West Africa. Crown 8vo. 8*s.* 6*d.*

A TREATISE ON HIGHER TRIGONOMETRY. Globe 8vo. 4*s.* 6*d.*

A TREATISE ON ELEMENTARY TRIGONOMETRY AND HIGHER TRIGONOMETRY. Extra Fcap. 8vo. 7*s.* 6*d.*

MACMILLAN AND CO., Ltd., LONDON.

DATE DUE SLIP

F. 255

QA 152 S64 1906 SOL-
SMITH CHARLES 1844-1916
ELEMENTARY ALGEBRA

39455022 CURR HIST

000008506719

QA 152 S64 1906 sol.
Smith, Charles, 1844-1916.
Elementary algebra

39455022 CURR HIST